Specker's Proof of Infinity in **NF**

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$\mathcal{L}_\in := \{=, \in\}$.

The **logic** is *Classical with Equality.*

Extensionality is an axiom

$$\textbf{Ext}: \qquad \forall x \forall y \left(\forall z (z \in x \leftrightarrow z \in y) \to x = y \right).$$

Definition 1 Stratification *of a formula* φ *is an assignment of natural numbers* (type indices) *to variables (both free and bound) in* φ *s.t. for atomic subformulas of* φ *only the following variants are allowed:*
(a) $x^i = y^i$*;*
(b) $x^i \in y^{i+1}$.
A formula φ *is* stratified *iff there exists a stratification of* φ*.*

Equivalently, a formula is stratified *iff it can be obtained from a formula of Simple Type Theory by erasing type indices (and renaming variables if necessary).*

Examples. The formula $x \in y \wedge y \in z$ is stratified, but the formula $x \in y \wedge y \in x$ is not.

Stratified Comprehension is an axiom scheme

$$\mathbf{SCA}: \qquad \exists y \forall x \left(x \in y \leftrightarrow \varphi[x] \right),$$

for every stratified formula φ with y not free in φ.

$$\mathbf{NF} := \mathbf{SCA} + \mathbf{Ext}.$$

Known facts:

- $\mathrm{Consis}(\mathbf{NF} + \ldots) \to \mathrm{Consis}(\mathbf{ZF} + \ldots)$;
- $\mathbf{NF} \vdash \neg\mathbf{AC}$;
- $\mathbf{NF} \vdash \mathbf{Inf}$;
- $\mathbf{PA} \vdash \mathrm{Consis}(\mathbf{NF}_3)$;
- $\mathbf{NF} = \mathbf{NF}_4$;
- $\mathrm{Consis}(\mathbf{NFU}) \Leftrightarrow \mathrm{Consis}(\mathbf{I\Delta}_0 + \mathbf{Exp})$;
- ...

Main unknown question (since 1937):

- $\mathrm{Consis}(\mathbf{ZF} + \ldots) \to \mathrm{Consis}(\mathbf{NF})$?

Russell's Paradox is not derivable in **NF**, for $M := \{x \mid x \notin x\}$ cannot claimed to be a set. Nor any other known "paradox" goes through.

Axioms of **ZF**: 1908: *Extensionality, Pair, Union, Infinity, Separation, Powerset,*
1917: *Foundation*, 1922: *Replacement*

Which **ZFC** *axioms are provable in* **NF***?*

- *Extensionality:* $\vdash$.
- *Pair:* $\vdash$, $\forall a^{(0)}\,\forall b^{(0)}\,\exists z^{(1)}\,\forall x^{(0)}\,(x \in z \leftrightarrow x = a \vee x = b)$.
- *Union:* $\vdash$, $\forall a^{(2)}\,\exists z^{(1)}\,\forall x^{(0)}\,(x \in z \leftrightarrow \exists u^{(1)}\!\in\! a\, x \in u)$.
- *Powerset:* $\vdash$, $\forall a^{(1)}\,\exists z^{(2)}\,\forall x^{(1)}\,(x \in z \leftrightarrow \forall u^{(0)}\!\in\! x\, u \in a)$.
- *Infinity:* $\vdash$, very non-trivial proof, [Specker53].
- *Separation:* $\vdash$ *strat.*, $\forall a^{(1)}\,\exists z^{(1)}\,\forall x^{(0)}\,(x \in z \leftrightarrow x \in a \wedge \varphi[x])$ (small trick if $a \in \mathrm{FV}(\varphi)$).
  Non-strat.: Let $\mathrm{V} := \{x \mid x = x\}$. Then $\exists z \forall x\,(x \in z \leftrightarrow x \in \mathrm{V} \wedge x \notin x)$ yields Russell's Paradox. Therefore $\vdash \neg$*non-strat.*.
- *Replacement:* $\vdash$ *strat.*,
  $\forall a\,\big(\forall x\!\in\! a\,\exists! y \varphi[x,y] \rightarrow \exists z^{(1)}\,\forall y^{(0)}\,(y \in z \leftrightarrow \exists x^{(*)}\!\in\! a^{(*+1)}\,\varphi[x,y])\big)$ (the same small trick if $a \in \mathrm{FV}(\varphi)$).
  $\vdash \neg$*non-strat.*, requires work.
- *Foundation:* $\vdash \neg$, since $\mathrm{V} \in \mathrm{V}$. Requires work.
- *Choice:* $\vdash \neg$, [Specker53]. All "reasonable" forms of **AC** are OK.

So, much mathematics can be developed. Much "elementary" set theory can be developed in **NF** in a reasonably standard way. "Later", however, there are substantial differences.

$$\Lambda^{(1)} := \{x^{(0)} \mid x \neq x\}.$$

Thus, $\forall x\, x \notin \Lambda$.

$$\mathrm{V}^{(1)} := \{x^{(0)} \mid x = x\}.$$

Thus, $\forall x\, x \in \mathrm{V}$.

Theorem 2 *([Specker 62])*

1. **NF** *is consistent iff there is a model of* **TNT** *[***TST** *is fine] with a type-shifting automorphism [=: tsau]* σ*.*
2. **NF** *is equiconsistent with the Theory of Types,* **TNTA** *[***TSTA** *is fine] with the* Ambiguity *scheme,* **Amb***,*

$$\varphi \leftrightarrow \varphi^+,$$

   for all sentences φ*. [*φ^+ *is the result of raising all type indices in* φ *by 1.]*

3. *When* ψ *is a stratified sentence, then* $\mathbf{NF} + \psi$ *is equiconsistent with* $\mathbf{TNTA} + \psi^{min}$*. [*$\psi^{min}$ *is the minimal stratification of* ψ*.]*

Proof. See [6]. (1) If $\langle U_i, =_i, \in_i \rangle_{i \in \mathbb{Z}}$ is a model of **TNT** with a tsau σ, then $\langle U, =, \in \rangle$ with

$$U := U_0, \quad x = y :\Leftrightarrow x =_0 y, \quad x \in y :\Leftrightarrow x \in_0 \sigma(y)$$

is a model of **NF**. Conversely, if $\langle U, =, \in \rangle$ is a model of **NF**, then $\langle U, =, \in \rangle_{i \in \mathbb{Z}}$ is a model of **TNT** with a tsau $\sigma := \mathsf{id}$. □

Theorem 3 *([Grishin 69])* *There is a model of* $\mathbf{NF}_3$*.*

Theorem 4 *([Grishin 73])* $\mathbf{NF} = \mathbf{NF}_4$*. Thus, Consis*(**NF**) *is equivalent to* Consis($\mathbf{TSTA}_4$)*, the Type Theory with Ambiguity using types 0, 1, 2 and 3 only.*

Equivalently, one can try to build a model for $\mathbf{TST}_4$, with $(=, \in)$-isomorphisms between type domains... For $\mathbf{TST}_3$ it was done by [Grishin 73].

(Frege) natural numbers in NF

$$0^{(2)} := \{x^{(1)} \mid \forall y^{(0)}\, y \notin x\}^{(2)} = \{\Lambda\}.$$

$$S(n^{(2)}) := \{x^{(1)} \cup \{y^{(0)}\} \mid x \in n \wedge y \notin x\}^{(2)}.$$

Thus,

$$1 = \{x \cup \{y\} \,|\, x \in 0 \wedge y \notin x\} = \{\Lambda \cup \{y\} \,|\, y \notin \Lambda\} = \{\{y\}\}; \quad (1)$$

$$\begin{aligned} 2 &= \{x \cup \{y\} \mid x \in 1 \wedge y \notin x\} \\ &= \{x \cup \{y\} \mid \exists z\, (x = \{z\}) \wedge y \notin x\} \\ &= \{x \cup \{y\} \mid \exists z\, (x = \{z\} \wedge y \neq z)\} \\ &= \{\{z, y\} \mid z \neq y\}; \end{aligned} \quad (2)$$

$$3 = \{\{z, y, x\} \mid z \neq y \wedge y \neq x \wedge x \neq z\}; \quad (3)$$

etc.

— *n is "the set of all sets with exactly n elements".*

We can also define

$$\mathbb{N}^{(3)} := \bigcap \{X^{(3)} | 0^{(2)} \in X \wedge \forall n^{(2)}\, (n \in X \to S(n) \in X)\}^{(4)}. \quad (4)$$

From the Definition (4) we immediately have a

Theorem *(Mathematical Induction)* *If $X \subseteq \mathbb{N}$, $0 \in X$ and $\forall n\, (n \in X \to S(n) \in X)$, then $X = \mathbb{N}$.*

It looks like we've already implemented the whole of **PA** *in* **NF**, $\mathbb{N}$ *being the "infinite" set. Is it true??*

But how do we know that there are "infinitely many" distinct elements in V *(to make all natural numbers not* Λ*)? Have we checked all Peano axioms?*

Assume that V is "small", e.g. $\mathrm{V} \in 2$. Then, by (2), $\exists z \exists y\, (z \neq y \wedge \mathrm{V} = \{z, y\})$. But then, by (3),
$3 = \{\{z, y, x\} \mid z \neq y \wedge y \neq x \wedge x \neq z\} = \Lambda$!

Also, we have

$$S(\Lambda) = \{x \cup \{y\} \mid x \in \Lambda \wedge y \notin x\} = \Lambda.$$

So, we have $\Lambda = 3 = 4 = 5 = \ldots$, while $3 = \Lambda \neq 2$.
This situation clearly breaks injectivity of S!

All of the following theorems either follow immediately from the Definitions, or are proved by Mathematical Induction. See Holmes [3, pp. 84–85].

Theorem $0 \in \mathbb{N}$.

Theorem *If* $n \in \mathbb{N}$*, then* $S(n) \in \mathbb{N}$*.*

Theorem *If* $n \in \mathbb{N}$*, then* $S(n) \neq 0$*.*

Theorem *If* $n \in \mathbb{N}$ *and* $n \neq 0$*, then* $n = S(m)$ *for some* $m \in \mathbb{N}$*.*

Theorem *If* $\forall k \in \mathbb{N}\, k \neq \Lambda$*,* $n, m \in \mathbb{N}$ *and* $S(n) = S(m)$*, then* $n = m$*.*

Also, observe

Lemma A *If* $n = \Lambda$ *for some* $n \in \mathbb{N}$*, then* $\mathrm{V} \in m$ *for some* $m \in \mathbb{N}$*.*

Proof. Assume that $n = \Lambda$. Since $0 = \{\Lambda\} \neq \Lambda$, by Mathematical Induction $\exists m \in \mathbb{N}\, (m \neq \Lambda \wedge S(m) = \Lambda)$. Fix such an m. Since $m \neq \Lambda$, $\exists x_0\, x_0 \in m$. Fix such an x_0. We also have

$$S(m) = \{x \cup \{y\} \mid x \in m \wedge y \notin x\} = \Lambda.$$

This only can be if $\forall y\, y \in x_0$. By Extensionality, then, $x_0 = \mathrm{V}$, yielding $\mathrm{V} \in m$. □

Contrapositioning Lemma A, we obtain

Lemma B *If* $\forall m \in \mathbb{N}\, \mathrm{V} \notin m$*, then* $\forall n \in \mathbb{N}\, n \neq \Lambda$*.*

Thus, in order to obtain a faithful representation of **PA** *in* **NF***, it remains to prove* $\forall m \in \mathbb{N}\, \mathrm{V} \notin m$ *("*$\mathrm{V} \notin \mathrm{Fin}$*" in* **NF** *terminology). In the remainder we will do it by showing that* V *cannot be well-ordered.*

Wiener-Kuratowski *ordered pair* is defined in the standard way:

$$\langle x, y\rangle^{(2)} := \{\{x^{(0)}\}, \{x^{(0)}, y^{(0)}\}\},$$

as well as *relations*, *functions*, etc.

Also, "X is a well-ordering" is defined as usual, by a stratified formula $\mathrm{WO}(X)$:

$$X \text{ is a set of ordered pairs} \wedge \mathrm{LO}(X) \wedge \forall Y \subset \mathrm{dom}(X)$$
$$\big(Y \neq \Lambda \rightarrow \exists y \in Y \forall x \in Y\ \langle y, x\rangle \in X\big)\,.$$

Ordinal is a set of well-orderings s.t. btw domains of any two of them there is an order-preserving bijection. *Ordinal arithmetic* is developed in the standard way.

There is a set Ω of all well-orderings, ordered by $\leq$, which is also a w.o. So, there is the greatest ordinal. (Burali-Forti paradox is avoided due to the stratification problems.)

Cardinal is an equivalence class under equinumerosity (expressed by bijections). *Elementary* cardinal arithmetic can be developed as usual (avoiding **AC**).

$$\mathrm{WO}^*(X, Y) :\Leftrightarrow \mathrm{WO}(Y) \wedge \mathrm{dom}(Y) = X.$$

Provable by Math. Induction:

Theorem $\quad \forall n \in \mathbb{N} \forall x \in n \exists X\ \mathrm{WO}^*(x, X).$

BIG Theorem [Specker 53] $\neg\exists X\, \mathrm{WO}^*(\mathrm{V}, X)$.

Corollary 1 $\forall n{\in}\mathbb{N}\, \mathrm{V} \notin n$; $\forall n{\in}\mathbb{N}\, n \neq \Lambda$.

Corollary 2 **PA** *can be faithfully embedded in* **NF**.

Corollary 3 **TST+Inf** *and* $\mathbf{Z}^{\Delta_0}$ *can be faithfully embedded in* **NF**.

Proof (Solovay). We have to derive a contradiction in "**NF** + V can be well-ordered". By the Theorem 2.3, we will derive a contradiction in "**TNTA** + V_1 can be well-ordered".

Very briefly: *in the context of* **TNT**, *with its facts*

$$\mathcal{P}(\mathrm{V}_i) = \mathrm{V}_{i+1}$$

and

$$\|\mathrm{V}_{i+1}\| = 2^{\|\mathrm{V}_i\|},$$

the assumption "V_1 *can be well-ordered" contradicts* **Amb**.

Solovay exhibits the proof in the context of **ZFC**, to better communicate the main construction. In the end, everything should be done inside Type Theory (which can be done, is a lot of technical details, and was done so by [Specker 53] (inside **NF**)).

As usual, *cardinal* means the least ordinal of that cardinality.

Define a function G (a proper class) which maps the class **OR** of ordinals into the class of cardinals:

1) $G(0) = 0$;

2) $G(\alpha + 1) = 2^{G(\alpha)}$ (cardinal exponentiation);

3) if λ is a limit ordinal, then $G(\lambda)$ is $\sup\{G(\alpha) \mid \alpha < \lambda\}$.

Thus G restricted to the finite ordinals is the usual "stack of twos" function. And $G(\omega + \alpha) = \beth_\alpha$.

To each cardinal κ (which could be finite or infinite), we are going to assign the Specker invariant $\mathrm{Sp}(\kappa)$ which will be an integer in the set $\{0, 1, 2\}$.

Let then κ be a cardinal. Let λ be the least ordinal such that $G(\lambda) \geq \kappa$. Write $\lambda = \lambda_1 + n$ where λ_1 is a limit ordinal and $n \in \omega$. Then $\mathrm{Sp}(\kappa)$ is the residue of n mod 3.

The key fact is the following:

Proposition *Let κ be a cardinal. Let $\kappa_1 = 2^\kappa$ and let $\kappa_2 = 2^{\kappa_1}$. Then* $\mathrm{Sp}(\kappa_2)$ *is unequal to* $\mathrm{Sp}(\kappa)$*.*

/- Let λ be least such that $G(\lambda) \geq \kappa$. So $G(\lambda + 1) \geq \kappa_1$ and $G(\lambda + 2) \geq \kappa_2$. We claim that $G(\lambda) < \kappa_2$. It follows that the least ordinal, λ_2, such that $G(\lambda_2) \geq \kappa_2$ is either $\lambda + 1$ or $\lambda + 2$ from which the proposition follows.

In proving our claim there are three cases to consider:

Case 1: $\lambda = 0$. Then $G(\lambda) = 0 = \kappa < \kappa_2$.

Case 2: λ is limit. From the way G is defined at limits and $\forall \alpha {<} \lambda\, G(\alpha) < \kappa$, $G(\lambda) \leq \kappa$. So $G(\lambda) = \kappa < \kappa_2$.

Case 3: $\lambda = \beta + 1$ for some β. Then $G(\beta) < \kappa$. So $G(\lambda) \leq \kappa_1 < \kappa_2$. -/

Compressing the definition into Type Theory:

It is easy to find a sentence of the language of Type Theory that expresses "$\mathrm{Sp}(\|\mathrm{V}_0\|) = j$" (where $j \in \{0, 1, 2\}$). The amount of choice we need is at most "V_1 is well-orderable".

Specker's result that **AC** contradicts "typical ambiguity" now follows immediately from the proposition. Namely, in **TNT** + "V_1 is well-orderable" we have derived the sentence

$$\neg \left(\mathrm{Sp}(\|\mathrm{V}_0\|) = 0_0 \leftrightarrow \mathrm{Sp}(\|\mathrm{V}_2\|) = 0_2\right),$$

contradicting **Amb**.

$\square$

... As a result of all this, **PA** can be embedded in **NF** ...

Question (asked by T. Forster, R. Holmes, M. Rathjen, ...)
Specker's proof very essentially uses classical logic. Does **INF** *derive Infinity? Could* Consis(**INF**) *be easy to prove?* – Unknown.

References

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