Consistency of Strictly Impredicative **NF**

and a little more...

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$\mathcal{L}_{\in} := \{=, \in\}$.

Extensionality is an axiom

$$\mathbf{Ext}: \qquad \forall x \forall y \left(\forall z (z \in x \leftrightarrow z \in y) \to x = y \right).$$

Definition 1 Stratification *of a formula* φ *is an assignment of natural numbers* (type indices) *to variables (both free and bound) in* φ *s.t. for atomic subformulas of* φ *only the following variants are allowed:*
(a) $x^i = y^i$*;*
(b) $x^i \in y^{i+1}$.

A formula φ is stratified *iff there exists a stratification of φ.*

Equivalently, a formula is stratified *iff it can be obtained from a formula of Simple Type Theory by erasing type indices (and renaming variables if necessary).*

Examples. The formula $x \in y \wedge y \in z$ is stratified, but the formula $x \in y \wedge y \in x$ is not.

Stratified Comprehension is an axiom scheme

$$\mathbf{SCA}: \qquad \exists y \forall x \left(x \in y \leftrightarrow \varphi[x] \right),$$

for every stratified formula φ with y not free in φ.

$$\mathbf{NF} := \mathbf{SCA} + \mathbf{Ext}.$$

$\mathbf{V}$ does exist:

$$\mathbf{V} := \{x \mid x = x\}.$$

So, $\mathbf{V} \in \mathbf{V}$, $\mathbf{V} = \mathcal{P}(\mathbf{V})$, etc. *Foundation* fails, *Cantor's Theorem* fails, as well as many other **ZFC** theorems, too.

Known facts:

- $\mathrm{Consis}(\mathbf{NF} + \ldots) \to \mathrm{Consis}(\mathbf{ZF} + \ldots)$;
- $\mathbf{NF} \vdash \neg\mathbf{AC}$;
- $\mathbf{NF} \vdash \mathbf{Inf}$;
- $\mathbf{PA} \vdash \mathrm{Consis}(\mathbf{NF}_3)$;
- $\mathbf{NF} = \mathbf{NF}_4$;
- $\ldots$

Main unknown question (since 1937):

- Consis($\mathbf{ZF} + \ldots$) $\to$ Consis($\mathbf{NF}$) ?

[2] M. Crabbé. On the consistency of an impredicative subsystem of Quine's NF. *Journal of Symbolic Logic* 47, pp. 131–136, 1982.

Definition 2 *(Crabbé) An instance of Stratified Comprehension*

$$\mathbf{SCA}: \qquad \exists y \forall x \left(x \in y \leftrightarrow \varphi[x] \right), \tag{1}$$

is predicative *iff there is a stratification of (1) s.t. the indices of bound variables in* φ *are* $<$ type(y), *and the indices of free variables in* φ *are* $\leq$ type(y).

NFP is a subsystem of **NF** where **SCA** is restricted to predicative instances. **NFI** ("mildly impredicative") is an extension of **NFP** which allows bound variables in φ of types $\leq$ type(y).

Theorem 3 *([Crabbé 82]) Both* **NFP** *and* **NFI** *are consistent, where in addition*

$$|\mathbf{NFP}| < |\mathbf{EA}|,$$
$$|\mathbf{PA_2}| \leq |\mathbf{NFI}| < |\mathbf{PA_3}|.$$

Two kinds of proofs: *model-theoretic (countably saturated models)* and *proof-theoretic (cut-elimination).*

Theorem 4 *([Holmes 99])*

$$|\mathbf{NFI}| = |\mathbf{PA_2}|.$$

Consider the *Union* axiom:

$$\mathbf{U}: \qquad \forall z \exists y \forall x \left(x \in y \leftrightarrow \exists v \left(v \in z \wedge x \in v \right) \right). \tag{2}$$

Note that **U** is in **NF**, but not in **NFI**:

$$\forall z^2 \exists y^1 \forall x^0 \left(x \in y \leftrightarrow \exists v^1 \left(v \in z \wedge x \in v \right) \right).$$

Theorem 5 *([Crabbé 82])*

$$\mathbf{NFP} + \mathbf{U} = \mathbf{NFI} + \mathbf{U} = \mathbf{NF}.$$

Definition 6 *(S.T.)* *An instance of Stratified Comprehension*

$$\mathbf{SCA}: \qquad \exists y \forall x \left(x \in y \leftrightarrow \varphi[x] \right),$$

is strictly impredicative *iff there is a stratification of it s.t. the indices of all variables in φ are $\geq \mathrm{type}(y) - 1$.*

Let **NFSI** denote a subsystem of **NF** where **SCA** is restricted to strictly impredicative instances. Then:

Theorem 7 *(S.T., 08)* **NFSI** *(and a little more, e.g. existence of Frege natural numbers) is consistent, too.*

The proof uses a bit of Model Theory, and a lot of Set Theory (forcing).

Theorem 8 *([Specker 62])*

1. **NF** *is consistent iff there is a model of* **TNT** *[***TST** *is fine] with a type-shifting automorphism [=: tsau]* σ*.*

2. **NF** *is equiconsistent with the Theory of Types,* **TNTA** *[***TSTA** *is fine] with the* Ambiguity *scheme,* **Amb***,*

$$\varphi \leftrightarrow \varphi^+,$$

   for all sentences φ*. [*φ^+ *is the result of raising all type indices in* φ *by 1.]*

Proof. See [6]. $\square$

Specker's proof generalizes immediately to subsystems of **NF** where **SCA** is restricted. For **NFSI**, an equivalent Type Theory is **Ext** plus **Amb** plus all instances of

$$\exists y^{i+1} \forall x^i \left(x \in y \leftrightarrow \varphi[x] \right),$$

where all indices in φ are $\geq i$.

From the outset, we assume consistency of **ZFC**. Let $\langle M, \in\rangle$ be an Ehrenfeucht-Mostowski model of $\mathbf{ZF} + \mathbf{V} = \mathbf{L}$, i.e. a countable model with a non-trivial external $\in$-automorphism σ. W.l.o.g.w.m.a. that σ moves up at least one regular cardinal κ (in the sense of M):

In M, sets can be enumerated by ordinals, i.e. there is a formula $\varphi(x, \alpha)$ s.t. the sentence "φ gives a (class) bijection between $\mathbf{V}$ and $\mathbf{On}$" is true in M. By Ehrenfeucht-Mostowski, $\sigma(x) \neq x$ for some $x \in M$. Since we have a definable bijection, $\sigma(\alpha) \neq \alpha$ for some ordinal $\alpha \in M$. If $\alpha < \sigma(\alpha)$, fine; if not, take σ^{-1}.
In order to move up a cardinal, use a definable bijection $\alpha \mapsto \aleph_\alpha$.
In order to move up a regular cardinal, use a definable injection $\alpha \mapsto \aleph_{\alpha+1}$.

By default, we will use forcing machinery (original results due to P. Cohen and R. Solovay) as laid out in

[5] K. Kunen. **Set Theory. An Introduction to Independence Proofs.** Elsevier, 1980.

Given a finite set S of **TSTA**-axioms, let $n \geq 2$ be such that all indices i in S fall under $0 \leq i \leq n$. For $0 \leq i < n$, let $\mathbb{P}_i := \mathrm{Fn}(\sigma^{i+1}(\kappa), 2, \sigma^i(\kappa))$ (Cohen's poset), where

$$\mathrm{Fn}(\kappa_1, 2, \kappa_0) := \{p \,|\, |p| < \kappa_0 \wedge p \text{ is a function} \wedge \mathrm{dom}(p) \subset \kappa_1 \wedge \mathrm{ran}(p) \subset 2\} \tag{3}$$

(see VII 6.1), and $\mathbb{P} := \mathbb{P}^n := \prod_{0 \leq i < n} \mathbb{P}_i$.

Note first that σ acts as a bijection between $\sigma^i(\kappa)$ and $\sigma^{i+1}(\kappa)$.

Let G_0 be $\mathbb{P}_0$-generic over M. Then

$$M[G_0] \models \exists h_0 \, h_0 : \sigma(\kappa) \overset{\text{bi}}{\mapsto} \mathcal{P}(\kappa).$$

Definition 9

$$\mathcal{P}_{<\omega}(b) := \{a \subset b \mid |a| < \omega\}.$$

Let $g_0 \in M$ be such that

$$g_0 : \kappa \overset{\text{bi}}{\mapsto} \mathcal{P}_{<\omega}(\kappa).$$

Defining $g_i := \sigma^i(g_0)$, we get

$$g_i : \sigma^i(\kappa) \overset{\text{bi}}{\mapsto} \mathcal{P}_{<\omega}(\sigma^i(\kappa)). \tag{4}$$

Lemma 10 *Given* $M[G_0] \ni h_0 : \sigma(\kappa) \overset{\text{bi}}{\mapsto} \mathcal{P}(\kappa)$ *and* $M \ni g_0 : \kappa \overset{\text{bi}}{\mapsto} \mathcal{P}_{<\omega}(\kappa)$*, there exists a bijection* $M[G_0] \ni f_0 : \sigma(\kappa) \overset{\text{bi}}{\mapsto} \mathcal{P}(\kappa)$ *satisfying* $f_0 \restriction \kappa = g_0$*.*

Proof. Work in $M[G_0]$. Since $|\mathcal{P}(\kappa)| = \sigma(\kappa)$, $|\mathcal{P}_{<\omega}(\kappa)| = \kappa$ and $\mathcal{P}(\kappa) = \mathcal{P}_{<\omega}(\kappa) \bigcup \mathcal{P}_{\geq\omega}(\kappa)$, we must have $|\mathcal{P}_{\geq\omega}(\kappa)| = \sigma(\kappa)$, i.e. there is a bijection h_1 between $\sigma(\kappa)$ and $\mathcal{P}_{\geq\omega}(\kappa)$. Now, for $a \in \mathcal{P}(\kappa)$, define $f_0'(a)$ by

$$f_0'(a) := \begin{cases} g_0^{-1}(a) & \text{if } a \in \mathcal{P}_{<\omega}(\kappa), \\ \kappa + h_1^{-1}(a) & \text{otherwise.} \end{cases} \tag{5}$$

We claim that f_0' is a special bijection between $\mathcal{P}(\kappa)$ and $\sigma(\kappa)$:

(i) $f_0'(a) < \sigma(\kappa)$ is seen from (5) and the fact that $\sigma(\kappa)$ is an additive principal number, i.e. an ordinal closed under ordinal sum;

(ii) f_0' *is onto:* if $\alpha < \kappa$, then by the first line of (5) $f_0'(a) = g_0^{-1}(a) = \alpha$ for some $a \in \mathcal{P}_{<\omega}(\kappa)$; otherwise, $\alpha = \kappa + \beta$ for some $\beta < \sigma(\kappa)$, and then $f_0'(a) = \kappa + h_1^{-1}(a)$ for some $a \in \mathcal{P}_{\geq\omega}(\kappa)$;

(iii) f_0' *is 1-1* follows from (5) and the fact that both g_0^{-1} and h_1^{-1} are 1-1;

(iv) *further,* from the first line of (5) we have $f_0' \restriction \mathcal{P}_{<\omega}(\kappa) = g_0^{-1}$.

From (i-iv) above, f_0 can be taken as the inverse of f_0'.

□

Choose $f_0 : \sigma(\kappa) \overset{\text{bi}}{\mapsto} \mathcal{P}(\kappa)$ as guaranteed by Lemma 10.

Let $\tau \in M^{\mathbb{P}_0}$ be a name for f_0, so that

$$M[G_0] \models \tau_{G_0} : \sigma(\kappa) \overset{\text{bi}}{\mapsto} \mathcal{P}(\kappa). \tag{6}$$

By the Forcing Theorem VII 3.6

$$\exists p \in G_0 \left(p \Vdash^{*}_{\mathbb{P}_0} \tau : (\sigma(\kappa))\check{}_{\mathbb{P}_0} \overset{\text{bi}}{\mapsto} \mathcal{P}((\kappa)\check{}_{\mathbb{P}_0}) \right)^M . \tag{7}$$

Taking $p \in G_0$ from (7) and applying σ^i to this formula, we obtain

$$\left(\sigma^i(p) \Vdash^{*}_{\mathbb{P}_i} \sigma^i(\tau) : (\sigma^{i+1}(\kappa))\check{}_{\mathbb{P}_i} \overset{\text{bi}}{\mapsto} \mathcal{P}((\sigma^i(\kappa))\check{}_{\mathbb{P}_i}) \right)^M . \tag{8}$$

Define $G_{i+1} := \sigma'' G_i$, $0 \leq i < n-1$, and $G := \prod_{0 \leq i < n} G_i$. Then each G_i contains $\sigma^i(p)$ and is $\mathbb{P}_i$-generic over M – see Lemma 11. It's easily verified that G is a filter on $\mathbb{P} = \prod_{0 \leq i < n} \mathbb{P}_i$, but it was more of an issue whether G is generic. Also observe that $\sigma^i(\tau) \in M^{\mathbb{P}_i}$, for each i.

Lemma 11 *(See pp. 219–220)*

$$\mathbb{G} \textit{ is } \mathbb{P}\textit{-generic over } M \iff \sigma''\mathbb{G} \textit{ is } \sigma(\mathbb{P})\textit{-generic over } M.$$

Proof. "$\mathbb{G}$ is a filter in $\mathbb{P}$" being equivalent to "$\sigma''\mathbb{G}$ is a filter in $\sigma(\mathbb{P})$" follows from σ being an isomorphism between $\mathbb{P}$ and $\sigma(\mathbb{P})$. For the "generic" part, it follows from "D is dense in $\mathbb{P}$" $\Leftrightarrow$ "$\sigma''D$ is dense in $\sigma(\mathbb{P})$" (σ isomorphism) *and* $\sigma''D = \sigma(D)$ (σ $\in$-automorphism of M). $\square$

Starting with the complete embeddings $\mathbb{P}_i \mapsto \prod_{0\le i<n} \mathbb{P}_i$, define natural embeddings $\imath_i : M^{\mathbb{P}_i} \mapsto M^{\mathbb{P}}$ as in VII 7.12.

Lemma 12 *For each i, $0 \le i < n$, $M[G_i]$ is a transitive submodel of $M[G]$.*

Proof. Let $x \in M[G_i]$. Then $x = \rho_{G_i}$ for $\rho \in M^{\mathbb{P}_i}$. Then $\imath_i(\rho) \in M^{\mathbb{P}}$ and $x = \rho_{G_i} = (\imath_i(\rho))_G$ by VII 7.13(a), so that $x \in M[G]$.

Now, assume $x = \rho_{G_i} = (\imath_i(\rho))_G \in M[G_i]$, $y = \tau_G \in M[G]$, $y \in_{M[G]} x$. We need to show $y \in M[G_i]$ and $y \in_{M[G_i]} x$. We compute:

$$\begin{aligned} y \in_{M[G]} x &\iff \tau_G \in_{M[G]} (\imath_i(\rho))_G \\ &\iff \exists p {\in} G\, (\langle \tau, p\rangle \in \imath_i(\rho)) \\ &\overset{\text{VII 7.12}}{\iff} \exists p{\in} G\, \exists p_i {\in} G_i\, \exists \delta {\in} M^{\mathbb{P}_i}\, (\langle \delta, p_i\rangle \in \rho \wedge \tau = \imath_i(\delta) \\ &\qquad\qquad \wedge\, p = \langle \emptyset, \ldots, p_i, \ldots, \emptyset\rangle) \\ &\implies y = \delta_{G_i} \in M[G_i] \wedge \delta_{G_i} \in_{M[G_i]} \rho_{G_i}. \end{aligned}$$

$\square$

See Picture 1.

Interpret variables x^i of $\mathcal{L}_{\mathbf{TST}^n}$ as $x {\in} \sigma^i(\kappa)$, and interpret $x^i \in^i y^{i+1}$ as $x \in (\sigma^i(\tau))_{G_i}(y)$. First note that from (8) we have

$$M[G_i] \models (\sigma^i(\tau))_{G_i} : \sigma^{i+1}(\kappa) \overset{\text{bi}}{\mapsto} \mathcal{P}(\sigma^i(\kappa)), \tag{9}$$

for each $0 \le i < n$. For brevity, we denote

$$f_i := (\sigma^i(\tau))_{G_i} \in M[G_i] \overset{\text{L. 12}}{\subset} M[G].$$

From Lemma 10, we have

$$f_i{\restriction}\sigma^i(\kappa) = g_i. \tag{10}$$

We want to show that, under this interpretation, each axiom of $\mathbf{TSTA}^n$ is true in $M[G]$.

Let's check *Extensionality*. We don't claim (yet) that $M[G] \models$ **ZFC**, but at least we have

Lemma 13

$$M[G] \models \textit{Extensionality}.$$

Proof. Follows from VII 2.14. $\square$

<u>Remark</u>. If G happened to be generic, then $M[G]$ would be a model of **ZFC**.

We have to model

$$\forall x^{i+1} \forall y^{i+1} \left(\forall z^i (z \in^i x \leftrightarrow z \in^i y) \rightarrow x = y \right),$$

i.e. to prove, in $M[G]$,

$$\forall x{\in}\sigma^{i+1}(\kappa) \forall y{\in}\sigma^{i+1}(\kappa) \left(\forall z{\in}\sigma^i(\kappa)(z \in f_i(x) \leftrightarrow z \in f_i(y)) \rightarrow x = y \right). \tag{11}$$

Fix $x, y \in \sigma^{i+1}(\kappa)$, and assume

$$\forall z{\in}\sigma^i(\kappa)(z \in f_i(x) \leftrightarrow z \in f_i(y)). \tag{12}$$

Since $(f_i(x), f_i(y) \in \mathcal{P}(\sigma^i(\kappa)))^{M[G_i]}$, we have $(f_i(x), f_i(y) \subset \sigma^i(\kappa))^{M[G_i]}$, so by absoluteness $f_i(x), f_i(y) \subset \sigma^i(\kappa)$. Then, (12) can be reduced to

$$\forall z \, (z \in f_i(x) \leftrightarrow z \in f_i(y)), \tag{13}$$

which implies

$$f_i(x) = f_i(y)$$

by *Extensionality* of $M[G]$. Now, since the functions f_i are 1-1 in $M[G_i]$ (see (9)), by absoluteness they are 1-1 in $M[G]$, so we can conclude $x = y$.

For *Ambiguity*, it's enough to model

$$\forall x^i \forall y^{i+1} \left(x \in^i y \leftrightarrow \sigma(x) \in^{i+1} \sigma(y) \right),$$

i.e. to have, in $M[G]$,

$$\forall x{\in}\sigma^i(\kappa) \forall y{\in}\sigma^{i+1}(\kappa) \left(x \in f_i(y) \leftrightarrow \sigma(x) \in f_{i+1}(\sigma(y)) \right), \quad (14)$$

$0 \leq i < n-1$.

Lemma 14 *For each i, $0 \leq i < n-1$, there is an $\in$-isomorphism σ_i of $M[G_i]$ onto $M[G_{i+1}]$ extending $\sigma \restriction M$; additionally, $\sigma_i(f_i) = f_{i+1}$.*

Proof. See Lemma 15 – Corollary 19. □

Coming back to (14), fix $x \in \sigma^i(\kappa)$, $y \in \sigma^{i+1}(\kappa)$. Assume $M[G] \models x \in f_i(y)$ (the opposite direction being analogous). Since the formula "$x \in f_i(y)$" is Δ_0, by absoluteness

$$M[G_i] \models x \in f_i(y).$$

By Lemma 14,

$$M[G_{i+1}] \models \sigma(x) \in f_{i+1}(\sigma(y)).$$

By absoluteness again, $M[G] \models \sigma(x) \in f_{i+1}(\sigma(y))$.

Lemma 15 *(See p. 222)* $\sigma : M^{\mathbb{P}} \overset{\text{bi}}{\mapsto} M^{\sigma(\mathbb{P})}$ *and* σ *is an* $\in$-*isomorphism between* $\langle M^{\mathbb{P}} \times \mathbb{P}, M^{\mathbb{P}} \rangle$ *and* $\langle M^{\sigma(\mathbb{P})} \times \sigma(\mathbb{P}), M^{\sigma(\mathbb{P})} \rangle$ *in the sense that for every* $\mu, \tau \in M^{\mathbb{P}}$ *and* $p \in \mathbb{P}$,

$$\langle \mu, p \rangle \in \tau \leftrightarrow \langle \sigma(\mu), \sigma(p) \rangle \in \sigma(\tau).$$

Proof. $\sigma : M^{\mathbb{P}} \overset{\text{bi}}{\mapsto} M^{\sigma(\mathbb{P})}$ follows from the fact that "$\tau \in M^{\mathbb{P}}$" is a formula of set theory with parameters $\tau, \mathbb{P}$. $\in$ is preserved since σ is an $\in$-automorphism. □

Definition 16

$$\Sigma \ = \ \{\langle \tau_{\mathbb{G}}, (\sigma(\tau))_{\sigma''\mathbb{G}} \rangle \mid \tau \in M^{\mathbb{P}}\}.$$

Lemma 17 Σ *is an* $\in$-*isomorphism between* $M[\mathbb{G}]$ *and* $M[\sigma''\mathbb{G}]$.

Proof (sketch). We need to check four things: (a) Σ is a function; (b) Σ is onto; (c) Σ is 1-1; (d) Σ commutes with $\in$. (b) follows from the fact that $\sigma : M^{\mathbb{P}} \mapsto M^{\sigma(\mathbb{P})}$ is onto, Lemma 15. (d): Let $\mu_{\mathbb{G}} \in \tau_{\mathbb{G}}$. Then $\exists p \in \mathbb{G}\, \langle \mu, p \rangle \in \tau$. Then $\sigma(p) \in \sigma''\mathbb{G}$ and $\langle \sigma(\mu), \sigma(p) \rangle \in \sigma(\tau)$ (Lemma 15). This means $(\sigma(\mu))_{\sigma''\mathbb{G}} \in (\sigma(\tau))_{\sigma''\mathbb{G}}$. (a) Let $\tau_{\mathbb{G}} = \tau'_{\mathbb{G}}$. Then $\forall \mu \in M^{\mathbb{P}}\, (\mu_{\mathbb{G}} \in \tau_{\mathbb{G}} \leftrightarrow \mu_{\mathbb{G}} \in \tau'_{\mathbb{G}})$. By (d) $\forall \mu \in M^{\mathbb{P}}\, ((\sigma(\mu))_{\sigma''\mathbb{G}} \in (\sigma(\tau))_{\sigma''\mathbb{G}} \leftrightarrow (\sigma(\mu))_{\sigma''\mathbb{G}} \in (\sigma(\tau'))_{\sigma''\mathbb{G}})$. By Lemma 15 this implies $\forall \mu \in M^{\sigma(\mathbb{P})}\, (\mu_{\sigma''\mathbb{G}} \in (\sigma(\tau))_{\sigma''\mathbb{G}} \leftrightarrow \mu_{\sigma''\mathbb{G}} \in (\sigma(\tau'))_{\sigma''\mathbb{G}})$, i.e. $(\sigma(\tau))_{\sigma''\mathbb{G}} = (\sigma(\tau'))_{\sigma''\mathbb{G}}$. (c) is analogous to (a): Σ^{-1} is a function. □

Lemma 18 *For every* $x \in M$, $\check{(x)}_{\mathbb{P}} = \check{(x)}_{\sigma(\mathbb{P})}$ *and* $\Sigma(x) = \sigma(x)$.

Proof. Since $1_{\mathbb{P}} = 1_{\sigma(\mathbb{P})} = \emptyset$, $(x)\check{}_{\mathbb{P}} = (x)\check{}_{\sigma(\mathbb{P})}$ is proved by induction on x. By Definition 16,

$$\Sigma(x) = \Sigma((\check{x})_{\mathbb{G}}) \overset{\text{Def. 16}}{=} (\sigma(\check{x}))_{\sigma''\mathbb{G}} \overset{\sigma\ \in\text{-auto}}{=} ((\sigma(x))\check{}\,)_{i''\mathbb{G}} = \sigma(x).$$

□

Corollary 19

$$\Sigma{\restriction}M = \sigma.$$

For *Comprehension*, we want to model

$$\forall x_1^{i_1} \dots \forall x_k^{i_k} \exists y^{i+1} \forall x^i \left(x \in^i y \leftrightarrow \varphi(x, x_1, \dots, x_k) \right).$$

This means to prove, in $M[G]$,

$$\begin{gathered}\forall x_1 \in \sigma^{i_1}(\kappa) \dots \forall x_k \in \sigma^{i_k}(\kappa) \exists y \in \sigma^{i+1}(\kappa) \forall x \in \sigma^i(\kappa) \\ \left(x \in f_i(y) \leftrightarrow \tilde{\varphi}(x, x_1, \dots, x_k, \sigma^{\iota_1}(\kappa), \dots, \sigma^{\iota_\ell}(\kappa), f_{j_1}, \dots, f_{j_l}) \right),\end{gathered} \tag{15}$$

where $\tilde{\varphi}$ is a translation of φ by the rules above.

Here I have a problem. It's unlikely that $M[G]$ satisfies (15) for every φ. One trivial result is immediate however from what stands: Consis($\mathbf{NF}_2$). *In that case in (15) $i = j_1 = \dots = j_l$, and the set*

$$A := \{x \in \sigma^i(\kappa) \mid \tilde{\varphi}(x, x_1, \dots, x_k, \sigma^{\iota_1}(\kappa), \dots, \sigma^{\iota_\ell}(\kappa), f_{j_1}, \dots, f_{j_l})\} \tag{16}$$

is in $M[G_i]$ by Separation; thus, also having $A \subset \sigma^i(\kappa)$, y can be taken to be $f_i^{-1}(A)$.

—

In general, everything boils down to showing $A \in M[G_i]$, but if G is not generic, it's even not clear whether A exists as a set in $M[G]$.

—

Assume in addition that G is $\mathbb{P}$-generic over M. Let an instance

$$\forall x_1 \ldots \forall x_k \exists y \forall x \left(x \in y \leftrightarrow \varphi(x, x_1, \ldots, x_k)\right) \tag{17}$$

of *Stratified Comprehension* be *strictly impredicative.* In that case, under our interpretation, (17) is true in $M[G]$. Indeed, it's enough to check only the case $i = 0$, for the general case follows then by *Ambiguity.* If $i = 0$, then $A \in M[G]$ since $M[G] \models$ *Separation*, and actually $A \in M[G_0]$ by VII 6.14 *(forcing above doesn't add subsets of smaller cardinals).*

—

Can do more, even without assuming G being generic or an axiom strictly impredicative. For example, the axioms

P7:	$\forall u \exists v \forall x \forall y \, (\langle y, x\rangle \in u \longleftrightarrow \langle x, y\rangle \in v)$	4	p.,∼s.i.
P8:	$\exists v \forall x \, (x \in v \longleftrightarrow \exists y (x = \{y\}))$	3	p.

can be pushed trough utilizing our careful choice of the initial bijection f_0:

(Frege natural numbers)

P8.k is an axiom

$$\exists v \forall x\,(x \in v \longleftrightarrow \exists y_1 \dots \exists y_k\,(\bigwedge_{1 \le \imath,\jmath \le k; \imath \ne \jmath} y_\imath \ne y_\jmath \wedge x = \{y_1, \dots, y_k\})). \tag{18}$$

((18) asserts existence of a Frege natural number $k \geq 1$. Note that (18) is predicative and not s.i. The reasoning below also works for $k = 0$, when we understand $\bigwedge_\emptyset$ as $\top$ and $\bigvee_\emptyset$ as $\bot$.)

That means that we must satisfy the following axiom of $\mathbf{TSTA}^n$:

$$\exists v^{i+2} \forall x^{i+1}\,(x \in v \longleftrightarrow \exists y^i_1 \dots \exists y^i_k\,(\bigwedge_{1 \le \imath,\jmath \le k; \imath \ne \jmath} y_\imath \ne y_\jmath \wedge x = \{y_1, \dots, y_k\})). \tag{19}$$

(Our formula $\varphi[x^{i+1}]$ in this case is "$\exists y^i_1 \dots \exists y^i_k\,(\bigwedge_{1 \le \imath,\jmath \le k; \imath \ne \jmath} y_\imath \neq y_\jmath \wedge x^{i+1} = \{y_1, \dots, y_k\})$".)

Given that

$$x^{i+1} = \{y^i_1, \dots, y^i_k\} \iff \bigwedge_{1 \le \imath \le k} y^i_\imath \in x^{i+1} \wedge \forall u^i {\in} x^{i+1} \bigvee_{1 \le \imath \le k} u = y^i_\imath, \tag{20}$$

we see that the translation of (20), for $y_1, \dots, y_k \in \sigma^i(\kappa)$, $x \in \sigma^{i+1}(\kappa)$, is

$$\bigwedge_{1 \le \imath \le k} y_\imath \in f_i(x) \wedge \forall u {\in} \sigma^i(\kappa)\,(u \in f_i(x) \rightarrow \bigvee_{1 \le \imath \le k} u = y_\imath),$$

i.e.

$$f_i(x) = \{y_1, \dots, y_k\}, \tag{21}$$

and in this case

$$\tilde{\varphi}(x, f_i) \;:=\; \exists y_1 {\in} \sigma^i(\kappa) \dots \exists y_k {\in} \sigma^i(\kappa)\,(\bigwedge_{1 \le \imath,\jmath \le k; \imath \ne \jmath} y_\imath \ne y_\jmath \wedge f_i(x) = \{y_1, \dots, y_k\}).$$

According to p. 13, in order to verify P8.k under my interpretation, we must have

$$A8.k := \begin{array}{c} \{x \in \sigma^{i+1}(\kappa) \mid \exists y_1 \in \sigma^i(\kappa) \ldots \exists y_k \in \sigma^i(\kappa) \\ (\bigwedge_{1 \le \imath,\jmath \le k; \imath \ne \jmath} y_\imath \ne y_\jmath \wedge f_i(x) = \{y_1, \ldots, y_k\})\} \end{array} \in M[G_{i+1}]. \tag{22}$$

(Note that $A8.k \in M[G_i] \subset M[G]$ automatically, because $f_i \in M[G_i]$ and $M[G_i]$ satisfies *Separation*, but what we actually need is $A8.k \in M[G_{i+1}]$.) Can we arrange for this?

What helps here is our special choice of f_i's:

Claim 20 *For $y_1, \ldots, y_k \in \sigma^i(\kappa)$, $x \in \sigma^{i+1}(\kappa)$, $f_i(x) = \{y_1, \ldots, y_k\}$ is equivalent to $x \in \sigma^i(\kappa) \wedge g_i(x) = \{y_1, \ldots, y_k\}$.*

Proof. $\Leftarrow$: Immediate from (10).

$\Rightarrow$: Let $a = \{y_1, \ldots, y_k\} \in \mathcal{P}_{<\omega}(\sigma^i(\kappa))$. From (9), (4) and (10), $f_i^{-1}(b) = g_i^{-1}(b)$ for $b \in \mathcal{P}_{<\omega}(\sigma^i(\kappa))$ (f_i^{-1} enumerates $\mathcal{P}(\sigma^i(\kappa))$ in a special regular way). From $f_i(x) = a$ we have $x = f_i^{-1}(a) = g_i^{-1}(a) \in \sigma^i(\kappa)$ (see (4)), so we must have $x \in \sigma^i(\kappa) \wedge g_i(x) = a$.

$\square$

Summarizing (see (20)–(21)), we have proved

Lemma 21 *Under $y_1, \ldots, y_k \in \sigma^i(\kappa)$, $x \in \sigma^{i+1}(\kappa)$, $(x = \{y_1, \ldots, y_k\})$" is equivalent to a Δ_0 formula with parameters in M.*

—

Coming back to P8.k, the formula $\varphi_{8.k}[x^{i+1}]$ in this case is "$\exists y_1^i \ldots \exists y_k^i \, (\bigwedge_{1 \le \imath,\jmath \le k; \imath \ne \jmath} y_\imath \ne y_\jmath \wedge x^{i+1} = \{y_1, \ldots, y_k\})$", and we must check

$$\begin{aligned} A8.k &:= \{x \in \sigma^{i+1}(\kappa) \mid \tilde{\varphi}_{8.k}[x]\} \\ &= \{x \in \sigma^{i+1}(\kappa) \mid \exists y_1 {\in} \sigma^i(\kappa) \ldots \exists y_k {\in} \sigma^i(\kappa) \\ &\qquad\qquad (\bigwedge_{1 \le \imath,\jmath \le k; \imath \ne \jmath} y_\imath \ne y_\jmath \wedge (x = \{y_1, \ldots, y_k\})\tilde{\ })\} \end{aligned}$$

being in $M[G_{i+1}]$. Lemma 21 actually gives us more: $A8.k \in M \subset M[G_{i+1}]$.

$(G = \prod_{0 \le i < n} G_i$ is generic over $M)$

Lemma 22 *(see VII Exercise B5)* *Assume $A \in M$, $f : A \mapsto M$ and $f \in M[\mathbb{G}]$. Then there is a $B \in M$ such that $f : A \mapsto B$.*

Proof. Let $f = \tau_{\mathbb{G}}$. We have

$$\forall x \in A\, \exists! b \in M\, \langle x, b\rangle \in f;$$

this yields

$$\forall x \in A\, \exists! b \in M\, ((\mathrm{op}(\check{x}, \check{b}))_{\mathbb{G}} \in \tau_{\mathbb{G}})^{M[\mathbb{G}]}.$$

By VII 3.6

$$\forall x \in A\, \exists! b \in M\, \exists p \in \mathbb{G}\, (p \Vdash^* \mathrm{op}(\check{x}, \check{b}) \in \tau)^M,$$

implying

$$\forall x \in A\, \exists! b \in M\, \exists p \in \mathbb{P}\, (p \Vdash^* \mathrm{op}(\check{x}, \check{b}) \in \tau)^M,$$

i.e.

$$M \models \forall x \in A\, \exists! b\, \exists p \in \mathbb{P}\, p \Vdash^* \mathrm{op}(\check{x}, \check{b}) \in \tau.$$

By *Replacement* (in M)

$$M \models \exists B = \{b \mid \exists x \in A\, \exists p \in \mathbb{P}\, p \Vdash^* \mathrm{op}(\check{x}, \check{b}) \in \tau\}.$$

We need to show $f : A \mapsto B$. Let $x \in A$ and $b \in M$ be such that $(\langle x, b\rangle \in f)^{M[\mathbb{G}]}$. Then $((\mathrm{op}(\check{x}, \check{b}))_{\mathbb{G}} \in \tau_{\mathbb{G}})^{M[\mathbb{G}]}$, and, by VII 3.6, $\exists p \in \mathbb{P}\, (p \Vdash^* \mathrm{op}(\check{x}, \check{b}) \in \tau)^M$, i.e. $b \in B$.

□

Corollary 23 *(see VII Exercise B6)* *Assume $\mathbb{P} \in M$ and α is an ordinal of M. Then (1) $\Rightarrow$ (2), where*

(1) whenever $B \in M$, ${}^{\alpha}B \cap M = {}^{\alpha}B \cap M[\mathbb{G}]$;

(2) ${}^{\alpha}M \cap M = {}^{\alpha}M \cap M[\mathbb{G}]$.

Proof. Assume (1). ${}^{\alpha}M \cap M \subset {}^{\alpha}M \cap M[\mathbb{G}]$ is obvious, so we need to show the converse. If $f \in {}^{\alpha}M \cap M[\mathbb{G}]$, then by Lemma 22 there is a $B \in M$ s.t. $f \in {}^{\alpha}B \cap M[\mathbb{G}]$. By (1) we have $f \in {}^{\alpha}B \cap M$, and, by transitivity of M, $f \in {}^{\alpha}M \cap M$. $\square$

VII 6.12. Definition. *A poset* $\mathbb{P}$ *is* λ-closed *iff whenever* $\gamma < \lambda$ *and* $\{p_\xi \,|\, \xi < \gamma\}$ *is a decreasing sequence of elements of* $\mathbb{P}$ *(i.e.,* $\xi < \eta \to p_\xi \geq p_\eta$*), then*

$$\exists q \in \mathbb{P}\, \forall \xi < \gamma\; q \leq p_\xi.$$

Lemma 24 *Assume* $\mathbb{P}$ *is* λ*-closed and* $\mathbb{G}$ *is* $\mathbb{P}$*-generic over* M*. Assume* $y \in M[\mathbb{G}]$*,* $y \subset M$*,* $(|y| < \lambda)^{M[\mathbb{G}]}$*. Then* $y \in M$*.*

Proof. We have an $\alpha < \lambda$ and an $f : \alpha \overset{\text{bi}}{\mapsto} y$ with $f \in M[\mathbb{G}]$. By VII 6.14, (1) of Corollary 23 is satisfied; consequently, so is (2). $f \in {}^{\alpha}M$, so by (2) $f \in M$, and thus $y = \operatorname{ran}(f) \in M$. $\square$

Lemma 25 *Assume* $\mathbb{P}$ *is* λ*-closed and* $\mathbb{G}$ *is* $\mathbb{P}$*-generic over* M*. Assume* $\kappa_0 < \kappa_1$ *and* $\kappa_0 \leq \lambda$*. Then* $(\mathrm{Fn}(\kappa_1, 2, \kappa_0))^M = (\mathrm{Fn}(\kappa_1, 2, \kappa_0))^{M[\mathbb{G}]}$.

Proof. $(\mathrm{Fn}(\kappa_1, 2, \kappa_0))^M \subset (\mathrm{Fn}(\kappa_1, 2, \kappa_0))^{M[\mathbb{G}]}$ follows by absoluteness and $M \subset M[\mathbb{G}]$, so we need to show the converse. Let $p \in M[\mathbb{G}]$ and $(p \in \mathrm{Fn}(\kappa_1, 2, \kappa_0))^{M[\mathbb{G}]}$. $\forall z \in p\, \exists x \in \kappa_1\, \exists i \in 2\; z = \langle x, i \rangle$, so $p \subset M$. We have $(|p| < \kappa_0 \leq \lambda)^{M[\mathbb{G}]}$, so by Lemma 24 $p \in M$. A bijection $f \in M[\mathbb{G}]$ between some $\alpha < \kappa_0$ and p is actually in M by VII 6.14, so that $(p \in \mathrm{Fn}(\kappa_1, 2, \kappa_0))^M$. $\square$

Lemma 26 *Assume* $\mathbb{P}$ *is* λ*-closed and* $\mathbb{G}$ *is* $\mathbb{P}$*-generic over* M*. Assume* $\kappa_0 < \kappa_1$*,* $\kappa_0 < \lambda$*, and* $\mathbb{G}_0$ *is* $\mathrm{Fn}(\kappa_1, 2, \kappa_0)$*-generic over* M*. Then* $\mathbb{G}_0$ *is* $\mathrm{Fn}(\kappa_1, 2, \kappa_0)$*-generic over* $M[\mathbb{G}]$*.*

Proof. By VII 6.10, $\mathrm{Fn}(\kappa_1, 2, \kappa_0)$ has the $(2^{<\kappa_0})^+$-c.c. In $M[\mathbb{G}]$, $2^{<\kappa_0} = \kappa_0$, since $M \models$ **GCH** and by VII 6.14 $M[\mathbb{G}]$ doesn't change powersets of cardinals below λ. Therefore, in $M[\mathbb{G}]$, every $\mathrm{Fn}(\kappa_1, 2, \kappa_0)$-antichain has cardinality $\leq \kappa_0$.

Now let D be a dense subset of $\mathrm{Fn}(\kappa_1, 2, \kappa_0)$ lying in $M[\mathbb{G}]$. We must show that $\mathbb{G}_0$ meets D. By Zorn (applied in $M[\mathbb{G}]$) let A be a maximal antichain consisting of elements of D (see Lemma 29). Then A has cardinality at most κ_0. By Lemma 24 A lies in M. A is clearly a maximal $\mathrm{Fn}(\kappa_1, 2, \kappa_0)$-antichain in the sense of M. But $\mathbb{G}_0$ is M-generic. So $\mathbb{G}_0$ meets A (see VII Exercise A12, Lemma 30). Hence $\mathbb{G}_0$ meets D.

$\square$

Corollary 27 *Under the conditions of Lemma 26,* $\mathbb{G}_0 \times \mathbb{G}$ *is* $\mathrm{Fn}(\kappa_1, 2, \kappa_0) \times \mathbb{P}$*-generic over* M*.*

Proof. Use Product Lemma VIII 1.4. $\square$

Theorem 28 $G = \prod_{0 \leq i < n} G_i$ *is* $\mathbb{P} = \prod_{0 \leq i < n} \mathbb{P}_i$*-generic over* M*.*

Proof. By backwards induction on j we prove that $\prod_{j \leq i < n} G_i$ is $\prod_{j \leq i < n} \mathbb{P}_i$-generic over M. The claim is obvious for $j = n - 1$; so we assume it for j, $0 < j < n$, and try to prove it for $j - 1$. (We remind $\mathbb{P}_{j-1} = \mathrm{Fn}(\sigma^j(\kappa), 2, \sigma^{j-1}(\kappa))$.) By Corollary 27, it's enough to see that $\prod_{j \leq i < n} \mathbb{P}_i$ is $\sigma^j(\kappa)$-closed. Each $\mathbb{P}_i$ is $\sigma^i(\kappa)$-closed, see VII 6.13. It follows that each $\mathbb{P}_i$ is $\sigma^k(\kappa)$-closed if $i \geq k$, see VII 6.12. It follows that $\prod_{j \leq i < n} \mathbb{P}_i$ is $\sigma^j(\kappa)$-closed, see Jech [4, 15.12]. $\square$

Cf. VII Exercise A12.

Lemma 29 *Assume $D \subset \mathbb{P}$ is dense. There is a maximal antichain $A \subset \mathbb{P}$ s.t. $A \subset D$.*

Proof. Let

$$\mathbb{A} := \{B \subset D \mid B \text{ is an antichain}\}.$$

Order $\mathbb{A}$ by inclusion. Every chain in $\mathbb{A}$ has a supremum, namely its union, for if $p_1, p_2 \in \bigcup \mathbb{C}$, then $\exists B_1 \in \mathbb{C}\, p_1 \in B_1$ and $\exists B_2 \in \mathbb{C}\, p_2 \in B_2$; $B_1 \subset B_2 \vee B_2 \subset B_1$; w.l.o.g.w.m.a. $B_1 \subset B_2$, then $p_1, p_2 \in B_2$ and $p_1 \perp p_2$ since B_2 is an antichain; thus $\bigcup \mathbb{C}$ is an antichain $\subset D$, too. By Zorn's Lemma, $\mathbb{A}$ has a maximal element A.

We need to prove A is a maximal antichain, not only maximal in $\mathbb{A}$. Let $A \cup \{a\}$ be an antichain. Since D is dense, $\exists q \in D\, q \leq a$. $A \cup \{q\}$ is also an antichain, and $A \cup \{q\} \subset D$, so $q \in A$. If $a \notin A$, we have a contradiction with $A \cup \{a\}$ being an antichain, since $a \not\perp q$. Consequently, $a \in A$, and A is a maximal antichain.

$\square$

Lemma 30 *Let $\mathbb{P} \in M$ and $\mathbb{G} \subset \mathbb{P}$ be a filter. The following two conditions are equivalent:*
(1) $\mathbb{G}$ meets every $\mathbb{P}$-dense set which is in M (i.e., $\mathbb{G}$ is $\mathbb{P}$-generic over M);
(2) $\mathbb{G}$ meets every $\mathbb{P}$ maximal antichain which is in M.

Proof. (2)⇒(1) follows from Lemma 29. For (1)⇒(2), let $A \in M$ be a $\mathbb{P}$ maximal antichain. Set

$$D := \{p \in \mathbb{P} \mid \exists q \in A\, p \leq q\}. \tag{23}$$

Obviously, $D \in M$.

Claim. D is dense in $\mathbb{P}$.

/- Assume not, i.e.

$$\exists p \in \mathbb{P} \, \forall q \in D \, q \not\leq p. \tag{24}$$

Claim 1. $p \notin A$.

/- If $p \in A$, then by (23) $p \in D$, contradicting (24). So $p \notin A$. *Claim 1-/*

Claim 2. $A \cup \{p\}$ is a $\mathbb{P}$ antichain in M.

/- We are to show $\forall q \in A \, p \perp q$. Assume not, i.e.

$$\exists q \in A \, \exists r \in \mathbb{P} \, (r \leq p \wedge r \leq q).$$

Since $r \leq q$, by (23) $r \in D$. But this contradicts (24). *Claim 2-/*

Now we have a contradiction with A being a maximal antichain. *Claim-/*

By (1), $\exists p \in \mathbb{G} \cap D$. By (23), $\exists q \in A \, p \leq q$. Since $\mathbb{G}$ is a filter, $q \in \mathbb{G} \cap A$.

□

Theorem 31 *Strictly impredicative* **NF** *(and a little more) is consistent.*

Proof. Above. □

So, this much effort it has taken to prove consistency of this fragment of **NF**. *It remains to be seen how much effort it will take to prove consistency of the whole theory.*

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