System L syntax for sequent calculi

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(based on works of or with Guillaume Munch-Maccagnoni, and nourished by an on-going collaboration with Marcelo Fiore)

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I) A syntactic tool-box

for sequent calculus proofs

The basic kit

Consider the cut rule, classically presented as :

$$\frac{\Gamma_1 \vdash A, \Delta_1' \qquad \Gamma_2', A \vdash \Delta_2}{\Gamma_1, \Gamma_2' \vdash \Delta_1', \Delta_2}$$

But $\Delta_1 = A, \Delta_1'$ and $\Gamma_2 = \Gamma_2', A$ might have several copies of A. One needs to specify which A is *active* in both assumptions.

For term assignments to natural deduction proofs, one associates variables to the formulas in a sequent $\vdash \Gamma$. Here too, contexts are lists of typed variable declarations. In system L notation, we set :

$$\frac{\dfrac{c : (\Gamma_1 \vdash \alpha : A, \Delta_1')}{\Gamma_1 \vdash \mu\alpha.c : A \mid \Delta_1'} \qquad \dfrac{c' : (\Gamma_2', x : A \vdash \Delta_2)}{\Gamma_2' \mid \tilde{\mu}x.c' : A \vdash \Delta_2}}{\langle \mu\alpha.c \mid \tilde{\mu}x.c' \rangle : (\Gamma_1, \Gamma_2' \vdash \Delta_1', \Delta_2)}$$

(note that μ, $\tilde{\mu}$ are binding operators)

Different judgements

Therefore, we distinguish different kinds of judgements :

- commands $c : (\Gamma \vdash \Delta)$ with no active formula which under Curry-Howard (and head reduction) will read as machine states

- terms $\Gamma \vdash v : A \,|\, \Delta$ which under Curry-Howard will read as programs of type A

- contexts $\Gamma \,|\, e : A \vdash \Delta$ which under Curry-Howard read as contexts expecting to interact with a program of type A

In *focused* systems, we shall also have *value* and *covalue* judgements in which the active formula is moreover *under focus*.

In monolateral systems, considered first in this talk, the context (and co-value) judgements disappear (replaced with terms or values of the dual type).

Pattern-matching

Formulas are polarised according to the rules used to introduce their top connective : these rules are irreversible=positive or reversible=negative.

We shall use constructors for denoting the irreversible rules, and **structured** binding operations μ (and $\tilde{\mu}$ on the left of sequents in bilateral systems) for the reversible rules. The dual of an irreversible connective being reversible, this will lead to **"cut-elimination through pattern-matching"** :

Irreversible

$$\frac{\vdash t_1 : A_1 \,|\, \Delta_1 \quad \vdash t_2 : A_2 \,|\, \Delta_2}{\vdash (t_1, t_2) : A_1 \otimes A_2 \,|\, \Delta_1, \Delta_2}$$

Reversible

$$\frac{c : (\vdash x_1 : A_1, x_2 : A_2, \Delta)}{\vdash \mu(x_1, x_2).c : A_1 \parr A_2 \,|\, \Delta}$$

$$\langle (t_1, t_2) \,|\, \mu(x_1, x_2).c \rangle \to c[t_1/x_1, t_2/x_2]$$

What is "system L" ?

Summarising, we use "system L" ("L" for Gentzen's terminology of sequent calculus systems) for term assignment systems for sequent calculus presentations of various logical systems that share the following features :

- different kinds of judgements, that make explicit the notion of active formula (possibly under focus) and coercions between them. We have seen activation via μ and $\tilde{\mu}$. Deactivation is achieved via "cut with axiom" :

$$\frac{\Gamma \vdash v : A \,|\, \Delta \qquad \overline{|\, \alpha : A \vdash \alpha : A}}{\langle v \,|\, \alpha \rangle : (\Gamma, \vdash \alpha : A, \Delta)}$$

This is the only form of cut that will *not* be evaluated in our formalism.

- structured pattern-matching for reversible rules

The first feature was put forward in Curien-Herbelin's duality of computation paper (ICFP 2000).

II) Polarised Classical logic

Two-sided polarised classical sequent calculus

$$\frac{}{A \vdash A} \qquad \frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_2 \vdash A, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

$$\frac{\Gamma_1 \vdash A_1, \Delta_1 \quad \Gamma_2 \vdash A_2, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A_1 \otimes A_2, \Delta_1, \Delta_2} \qquad \frac{\Gamma \vdash A_1, \Delta}{\Gamma \vdash A_1 \oplus A_2, \Delta} \qquad \frac{\Gamma \vdash A_2, \Delta}{\Gamma \vdash A_1 \oplus A_2, \Delta}$$

$$\frac{\Gamma_1, A_1 \vdash \Delta_1 \quad \Gamma_2, A_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2, A_1 ⅋ A_2 \vdash \Delta_1, \Delta_2} \qquad \frac{\Gamma, A_1 \vdash \Delta}{\Gamma, A_1 \& A_2 \vdash \Delta} \qquad \frac{\Gamma, A_2 \vdash \Delta}{\Gamma, A_1 \& A_2 \vdash \Delta}$$

$$\frac{\Gamma \vdash A_1, A_2, \Delta}{\Gamma \vdash A_1 ⅋ A_2, \Delta} \qquad \frac{\Gamma \vdash A_1, \Delta \quad \Gamma \vdash A_2, \Delta}{\Gamma \vdash A_1 \& A_2, \Delta} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}$$

$$\frac{\Gamma, A_1, A_2 \vdash \Delta}{\Gamma, A_1 \otimes A_2 \vdash \Delta} \qquad \frac{\Gamma, A_1 \vdash \Delta \quad \Gamma, A_2 \vdash \Delta}{\Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}$$

Gentzen's classical sequent calculus

Classical non-polarised logic has only one conjunction and one disjunction :

$$A ::= Z \parallel A \wedge A \parallel A \vee A \parallel \neg A$$

Gentzen's system picks the irreversible rules for $\wedge$ and $\vee$ on the left and on the right (i.e. $\wedge$ right intro is $\otimes$ right intro, $\wedge$ left intro is $\&$ left intro,...).

But other choices could be posssible, for example the "all reversible" presentation (which leads to a cristal clear of completeness wrt to truth table semantics : exercise !).

Exercise : If $\Gamma \vdash \Delta$ is provable in your favourite presentation of classical sequent calculus, show that for *any* decoration of all formulas, replacing each $\wedge$ by either $\otimes$ or $\&$ (and similarly for $\vee$), the resulting sequent is provable in polarised classical logic.

From two-sided to one-sided

One transforms the explicit (involutive) negation into an implicit one (pushed to the atoms) = De Morgan duality (denoted here by overlining). Thus one moves from

$$\begin{array}{l} A ::= P \parallel N \\ P ::= X^+ \parallel A \otimes A \parallel A \oplus A \parallel \neg N \\ N ::= X^- \parallel A \parr A \parallel A \& A \parallel \neg P \end{array}$$

to

$$A ::= P \parallel N \qquad P ::= X \parallel A \otimes A \parallel A \oplus A \qquad N ::= \overline{X} \parallel A \parr A \parallel A \& A$$

by setting $X ::= X^+ \parallel \neg X^-$ and by translating formulas as follows :

$$(X^+)^\dagger = X^+ \quad (X^-)^\dagger = \overline{\neg X^-} \quad (\neg A)^\dagger = \overline{A^\dagger} \quad (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger \quad \ldots$$

(note in particular that $(\neg X^-)^\dagger = \neg X^-$).

Then, sequents $\Gamma \vdash \Delta$ can be folded into $\vdash \overline{\Gamma}, \Delta$.

One-sided polarised classical sequent calculus

$$\frac{}{\vdash A, \overline{A}} \qquad \frac{\vdash P, \Delta_1 \quad \vdash \overline{P}, \Delta_2}{\vdash \Delta_1, \Delta_2}$$

$$\frac{\vdash A_1, \Delta_1 \quad \vdash A_2, \Delta_2}{\vdash A_1 \otimes A_2, \Delta_1, \Delta_2} \qquad \frac{\vdash A_1, \Delta}{\vdash A_1 \oplus A_2, \Delta} \qquad \frac{\vdash A_2, \Delta}{\vdash A_1 \oplus A_2, \Delta}$$

$$\frac{\vdash A_1, A_2, \Delta}{\vdash A_1 \parr A_2, \Delta} \qquad \frac{\vdash A_1, \Delta \quad \vdash A_2, \Delta}{\vdash A_1 \& A_2, \Delta}$$

$$\frac{\vdash \Delta}{\vdash A, \Delta} \qquad \frac{\vdash A, A, \Delta}{\vdash A, \Delta}$$

Syntax for one-sided polarised classical logic

There are three kinds of judgements :

Commands	Positive terms	Negative terms
$c : (\vdash \Gamma)$	$\vdash t^+ : P \mid \Gamma$	$\vdash t^- : N \mid \Gamma$

Terms :

$c ::= \langle t^+ \mid t^- \rangle$ which one may also write if needed as $\langle t^- \mid t^+ \rangle$

$t ::= t^+ \parallel t^-$

$x ::= x^+ \parallel x^-$

$t^+ ::= x^+ \parallel \mu x^-.c \parallel (t_1, t_2) \parallel inl(t) \parallel inr(t)$

$t^- ::= x^- \parallel \mu x^+.c \parallel \mu(x_1, x_2).c \parallel \mu[inl(x_1).c_1, inr(x_2).c_2]$

Typing rules for one-sided polarised classical logic

Contexts Γ consist of declarations $x^+ : N$ and $x^- : P$:

$$\frac{}{\vdash x : A \,|\, x : \overline{A}} \qquad \frac{c : (\vdash x : A, \Gamma)}{\vdash \mu x.c : A \,|\, \Gamma}$$

$$\frac{\vdash t^+ : P \,|\, \Gamma \quad \vdash t^- : \overline{P} \,|\, \Delta}{\langle t^+ \,|\, t^- \rangle : (\vdash \Gamma, \Delta)} \qquad \frac{\vdash t_1 : A_1 \,|\, \Gamma \quad \vdash t_2 : A_2 \,|\, \Delta}{\vdash (t_1, t_2) : A_1 \otimes A_2 \,|\, \Gamma, \Delta} \qquad \frac{\vdash t_1 : A_1 \,|\, \Gamma}{\vdash inl(t_1) : A_1 \oplus A_2 \,|\, \Gamma}$$

$$\frac{c : (\vdash x_1 : A_1, x_2 : A_2, \Gamma)}{\vdash \mu(x_1, x_2).c : A_1 \parr A_2 \,|\, \Gamma} \qquad \frac{c_1 : (\vdash x_1 : A_1, \Gamma) \quad c_2 : (\vdash x_2 : A_2, \Gamma)}{\vdash \mu[inl(x_1).c_1, inr(x_2).c_2] : A_1 \& A_2 \,|\, \Gamma}$$

$$\frac{c : (\vdash \Gamma)}{c : (\vdash x : A, \Gamma)} \qquad \frac{c : (\vdash x_1 : A, x_2 : A, \Gamma)}{c[x/x_1, x/x_2] : (\vdash x : A, \Gamma)}$$

Illustrating activation and deactivation

The term decoration for

$$\dfrac{\dfrac{\vdash N \oplus P, A, B, \Gamma_1}{\vdash N \oplus P, A \parr B, \Gamma_1} \qquad \vdash M, \Gamma_2}{\vdash (N \oplus P) \otimes M, A \parr B, \Gamma_1, \Gamma_2}$$

is as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{c : (\vdash x : N \oplus P, y_1 : A, y_2 : B, \Gamma_1)}{\vdash \mu(y_1, y_2).c : A \parr B \mid x : N \oplus P, \Gamma_1}}{\langle y \mid \mu(y_1, y_2).c \rangle : (\vdash y : A \parr B, x : N \oplus P, \Gamma_1)}}{\vdash \mu x.\langle y \mid \mu(y_1, y_2).c \rangle : N \oplus P \mid y : A \parr B, \Gamma_1} \qquad \vdash t : M \mid \Gamma_2}{\vdash (\mu x.\langle y \mid \mu(y_1, y_2).c \rangle, t) : (N \oplus P) \otimes M \mid y : A \parr B, \Gamma_1, \Gamma_2}$$

Reduction rules for one-sided polarised classical logic

$$
\begin{array}{l}
\langle t^+ \mid \mu x^+.c\rangle \to c[t^+/x^+] \qquad (t^+ \neq \mu x^-.c_1) \\
\langle \mu x^-.c \mid t^-\rangle \to c[t^-/x^-] \\
\langle (t_1, t_2) \mid \mu(x_1, x_2).c\rangle \to c[t_1/x_1, t_2/x_2] \\
\langle inl(t_1) \mid \mu[inl(x_1).c_1, inr(x_2).c_2]\rangle \to c_1[t_1/x_1]
\end{array}
$$

Substitution accounts for commutative cuts

Lemma : If $c :\vdash x : A, \Gamma$, then the occurrences of x in c occur as deactivations : $c = C[\langle x \mid t\rangle]$.

The left hand side of the first and second computation rules codify a situation where one of the cut formulas has *not* been just introduced, and the reduction commutes the cut upwards on the right (resp. on the left) to the places where it was introduced, so that eventually logical cut rules such as the third or the fourth rule can be applied :

$$\begin{array}{c} \langle t_1^+ \mid \mu x^+.c\rangle = \langle t_1^+ \mid \mu x^+.C[\langle x^+ \mid t_2^-\rangle]\rangle \\ \downarrow \\ c[t_1^+/x^+] = C[\langle t_1^+ \mid t_2^-\rangle \end{array}$$

This commutation can be treated as progressive (explicit substitution) or as a 1 shot reduction (as in λ-calculus).

Syntax for two-sided polarised classical logic

One has in addition positive and negative contexts :

	Commands	$c : (\Gamma \vdash \Delta)$		
Positive terms	$\Gamma \vdash v^+ : P \mid \Delta$		Negative terms	$\Gamma \vdash v^- : N \mid \Delta$
Positive contexts	$\Gamma \vdash \; \mid \Delta$		Negative contexts	$\Gamma \vdash \; \mid \Delta$

Terms :

$$c ::= \langle v^+ \mid e^+ \rangle \parallel \langle v^- \mid e^- \rangle$$
$$t ::= t^+ \parallel t^- \qquad x ::= x^+ \parallel x^- \qquad \alpha = \alpha^+ \parallel \alpha^-$$
$$v^+ ::= x^+ \parallel \mu\alpha^+.c \parallel (v_1, v_2) \parallel inl(v) \parallel inr(v) \parallel (e^-)^\neg$$
$$t^- ::= x^- \parallel \mu\alpha^-.c \parallel \mu[\alpha_1, \alpha_2].c \parallel \mu[\alpha_1[fst].c_1, \alpha_2[snd].c_2] \parallel (e^+)^\neg$$
$$e^+ ::= \alpha^+ \parallel \tilde{\mu}x^-.c \parallel [e_1, e_2] \parallel e[fst] \parallel e[snd] \parallel (t^-)^\neg$$
$$e^- ::= \alpha^- \parallel \tilde{\mu}x^+.c \parallel \tilde{\mu}(x_1, x_2).c \parallel \tilde{\mu}[inl(x_1).c_1, inr(x_2).c_2] \parallel (t^+)^\neg$$

Typing rules for two-sided polarised classical logic

$$\frac{}{x : A \vdash x : A \,|} \qquad \frac{}{|\, \alpha : A \vdash \alpha : A}$$

$$\frac{\Gamma \vdash v : A \,|\, \Delta}{\Gamma \,|\, v^{\neg} : \neg A \vdash \Delta} \qquad \frac{\Gamma \,|\, e : A \vdash \Delta}{\Gamma \vdash e^{\neg} : \neg A \,|\, \Delta}$$

$$\frac{c : (\Gamma \vdash \alpha_1 : A_1, \alpha_2 : A_2, \Delta)}{\Gamma \vdash \mu[\alpha_1, \alpha_2].c : A_1 \parr A_2 \,|\, \Delta} \qquad \frac{c : (\Gamma, x_1 : A_1, x_2 : A_2 \vdash \Delta)}{\Gamma \,|\, \tilde{\mu}(x_1, x_2).c : A_1 \otimes A_2 \vdash \Delta}$$

etc. . .

Reduction rules for for two-sided polarised classical logic

$$
\begin{array}{ll}
\langle v^+ \mid \tilde{\mu} x^+ . c \rangle \to c[v^+/x^+] & (v^+ \neq \mu\alpha^+.c) \\
\langle \mu\alpha^- . c \mid e^- \rangle \to c[e^-/\alpha^-] & (e^- \neq \tilde{\mu} x^-.c) \\
\langle v^- \mid \tilde{\mu} x^- . c \rangle \to c[v^-/x^-] & \\
\langle \mu\alpha^+ . c \mid e^+ \rangle \to c[e^+/\alpha^+] & \\
\langle (v_1, v_2) \mid \tilde{\mu}(x_1, x_2).c \rangle \to c[v_1/x_1, v_2/x_2] & \\
\langle \mu[\alpha_1, \alpha_2].c \mid [e_1, e_2] \rangle \to c[e_1/\alpha_1, e_2/\alpha_2] & \\
\langle inl(v_1) \mid \tilde{\mu}[inl(x_1).c_1, inr(x_2).c_2] \rangle \to c_1[v_1/x_1] & \\
\langle \mu(\alpha_1[fst].c_1, \alpha_2[snd].c_2)) \mid e_1[fst] \rangle \to c_1[e_1/\alpha_1] & \\
\langle e^\neg \mid v^\neg \rangle \to \langle v \mid e \rangle &
\end{array}
$$

NB : In principle, one would have four choices to avoid the critical pairs, but the one here seems the most meaningful one in view of the focalising restriction.

III) Focalised systems

Focalisation

The restriction on the μx^+ rule suggests a global call-by-value regime for the substitution of positive terms. This is achieved by (we revert to one-sided for simplicity) :

adding a new typing judgement :

$$\text{Values} \qquad \vdash V^+ : P \,;\, \Delta$$

and restricting the syntax as follows :

$$
\begin{array}{l}
c ::= \langle t^+ \mid t^- \rangle \\
x ::= x^+ \parallel x^- \\
V^+ ::= x^+ \parallel (V_1, V_2) \parallel inl(V) \parallel inr(V) \\
V ::= V^+ \parallel t^- \\
t^+ ::= V^+ \parallel \mu x^-.c \\
t^- ::= x^- \parallel \mu x^+.c \parallel \mu(x_1, x_2).c \parallel \mu[inl(x_1).c_1, inr(x_2).c_2]
\end{array}
$$

Illustrating focalisation

A focalised proof

$$\dfrac{\dfrac{\vdash N \mid A \parr B, \Gamma_1}{\vdash N \oplus P \,;\, A \parr B, \Gamma_1} \qquad \vdash M \mid \Gamma_2}{\vdash (N \oplus P) \otimes M \,;\, A \parr B, \Gamma_1, \Gamma_2}$$

A non focalised proof

$$\dfrac{\dfrac{\vdash N \oplus P, A, B, \Gamma_1}{\vdash N \oplus P, A \parr B, \Gamma_1} \qquad \vdash M, \Gamma_2}{\vdash (N \oplus P) \otimes M, A \parr B, \Gamma_1, \Gamma_2}$$

Typing rules for one-sided focalised classical logic

($\vdash V : A \parallel \Gamma$ stands for either $\vdash V^+ : P \,;\, \Gamma$ or $\vdash t^- : N \mid \Gamma$)

$$\frac{}{\vdash x^+ : P \,;\, x^+ : \overline{P}} \qquad \frac{}{\vdash x^- : N \mid x^- : \overline{N}} \qquad \frac{\vdash t^+ : P \mid \Gamma \qquad \vdash t^- : \overline{P} \mid \Delta}{\langle t^+ \mid t^- \rangle : (\vdash \Gamma, \Delta)}$$

$$\frac{\vdash V^+ : P \,;\, \Gamma}{\vdash V^+ : P \mid \Gamma} \qquad \frac{c : (\vdash x : A, \Gamma)}{\vdash \mu x.c : A \mid \Gamma}$$

$$\frac{\vdash V_1 : A_1 \parallel \Gamma \qquad \vdash V_2 : A_2 \parallel \Delta}{\vdash (V_1, V_2) : A_1 \otimes A_2 \,;\, \Gamma, \Delta} \qquad \frac{\vdash V_1 : A_1 \parallel \Gamma}{\vdash inl(V_1) : A_1 \oplus A_2 \,;\, \Gamma}$$

$$\frac{c : (\vdash x_1 : A_1, x_2 : A_2, \Gamma)}{\vdash \mu(x_1, x_2).c : A_1 \parr A_2 \mid \Gamma} \qquad \frac{c_1 : (\vdash x_1 : A_1, \Gamma) \quad c_2 : (\vdash x_2 : A_2, \Gamma)}{\vdash \mu[inl(x_1).c_1, inr(x_2).c_2] : A_1 \& A_2 \mid \Gamma}$$

$$\frac{c : (\vdash \Gamma)}{c : (\vdash x : A, \Gamma)} \qquad \frac{c : (\vdash x_1 : A, x_2 : A, \Gamma)}{c[x/x_1, x/x_2] : (\vdash x : A, \Gamma)}$$

Reduction rules for one-sided focalised classical logic

$$
\begin{array}{l}
\langle V^+ \mid \mu x^+.c\rangle \to c[V^+/x^+] \\
\langle \mu x^-.c \mid t^-\rangle \to c[t^-/x^-] \\
\langle (V_1, V_2) \mid \mu(x_1, x_2).c\rangle \to c[V_1/x_1, V_2/x_2] \\
\langle inl(V_1) \mid \mu[inl(x_1).c_1, inr(x_2).c_2]\rangle \to c_1[V_1/x_1]
\end{array}
$$

Call-by-value regime for substitution of positive variables, call-by-name regime for substitution of negative variables.

In the two-sided setting, we have in addition call-by-value regime for substitution of negative continuation variables, and call-by-name regime for substitution of positive continuation variables.

Explicit polarisation

$$P ::= X \parallel P \otimes P \parallel P \oplus P \parallel {\downarrow} N \quad N ::= \overline{X} \parallel N \parr N \parallel N \& N \parallel {\uparrow} P \quad A ::= P \parallel N$$

(not only formulas, but now also connectives have polarities, i.e. the tensor takes two positives and returns a positive)

Illustrating explicit versus implicit

Implicit

$$\frac{\vdash N \mid \Gamma_1 \qquad \vdash P\,;\,\Gamma_2}{\vdash N \otimes P\,;\,\Gamma_1, \Gamma_2}$$

Explicit

$$\frac{\dfrac{\vdash N \mid \Gamma_1}{\vdash {\downarrow}N\,;\,\Gamma_1} \qquad \vdash P\,;\,\Gamma_2}{\vdash {\downarrow}N \otimes P\,;\,\Gamma_1, \Gamma_2}$$

Read (bottom-up) $\downarrow$ as marking explicitly the exit from the focalisation phase.

Syntax for one-sided explicit focalised classical logic

Terms :

$$
\begin{array}{l}
c ::= \langle t^+ \mid t^- \rangle \\
V^+ ::= x^+ \parallel (V_1^+, V_2^+) \parallel inl(V^+) \parallel inr(V^+) \parallel (t^-)^{\downarrow} \\
t^+ ::= V^+ \parallel \mu x^-.c \\
t^- ::= x^- \parallel \mu x^+.c \parallel \mu(x_1^+, x_2^+).c \parallel \mu[inl(x_1^+).c_1, inr(x_2^+).c_2] \parallel \mu(x^-)^{\downarrow}.c
\end{array}
$$

Typing rules for one-sided explicit focalised classical logic

$$\frac{}{\vdash x^{+} : P \,;\, x^{+} : \overline{P}} \qquad \frac{}{\vdash x^{-} : N \mid x^{-} : \overline{N}} \qquad \frac{\vdash t^{+} : P \mid \Gamma \quad \vdash t^{-} : \overline{P} \mid \Delta}{\langle t^{+} \mid t^{-} \rangle : (\vdash \Gamma, \Delta)}$$

$$\frac{\vdash V^{+} : P \,;\, \Gamma}{\vdash V^{+} : P \mid \Gamma} \qquad \frac{c : (\vdash x : A, \Gamma)}{\vdash \mu x.c : A \mid \Gamma} \qquad \frac{c : (\vdash \Gamma)}{c : (\vdash x^{+} : A, \Gamma)} \qquad \frac{c : (\vdash x_1^{+} : A, x_2^{+} : A, \Gamma)}{c[x_2^{+}/x_1^{+}] : (\vdash x_2^{+} : A, \Gamma)}$$

$$\frac{\vdash V_1^{+} : P_1 \,;\, \Gamma \qquad \vdash V_2^{+} : P_2 \,;\, \Delta}{\vdash (V_1^{+}, V_2^{+}) : P_1 \otimes P_2 \,;\, \Gamma, \Delta} \qquad \frac{\vdash V_1^{+} : P_1 \,;\, \Gamma}{\vdash inl(V_1^{+}) : P_1 \oplus P_2 \,;\, \Gamma}$$

$$\frac{c : (\vdash x_1^{+} : N_1, x_2^{+} : N_2, \Gamma)}{\vdash \mu(x_1^{+}, x_2^{+}).c : N_1 \invamp N_2 \mid \Gamma} \qquad \frac{c_1 : (\vdash x_1^{+} : N_1, \Gamma) \quad c_2 : (\vdash x_2^{+} : N_2, \Gamma)}{\vdash \mu[inl(x_1^{+}).c_1, inr(x_2^{+}).c_2] : N_1 \& N_2 \mid \Gamma}$$

$$\frac{\vdash t^{-} : N \mid \Gamma}{\vdash (t^{-})^{\downarrow} : \downarrow N \,;\, \Gamma} \qquad \frac{c : (\vdash x^{-} : P, \Gamma)}{\vdash \mu(x^{-})^{\downarrow}.c : \uparrow P \mid \Gamma}$$

Reduction rules for one-sided explicit focalised classical logic

$$\langle V^+ \mid \mu x^+.c \rangle \to c[V^+/x^+]$$
$$\langle \mu x^-.c \mid t^- \rangle \to c[t^-/x^-]$$
$$\langle (V_1^+, V_2^+) \mid \mu(x_1^+, x_2^+).c \rangle \to c[V_1^+/x_1^+, V_2^+/x_2^+]$$
$$\langle inl(V_1^+) \mid \mu[inl(x_1^+).c_1, inr(x_2^+).c_2] \rangle \to c_1[V_1^+/x_1^+]$$
$$\langle (t^-)^{\downarrow} \mid \mu(x^-)^{\downarrow}.c \rangle \to c[t^-/x^-]$$

Weakly focalised classical logic

This system is essentially (an explicit version of) Girard's LC (Π is a set consisting of at most one positive formula). From now on, relax, we give up on proof-term syntax (the syntax for weakly focalised systems is still under elaboration)

$$\frac{}{\vdash P\,;\,\overline{P}} \qquad \frac{\vdash P\,;\,\Gamma \quad \vdash [Q]\,;\,\overline{P},\Delta}{\vdash [Q]\,;\,\Gamma,\Delta} \qquad \frac{\vdash [Q]\,;\,P,\Gamma \quad \vdash\,;\,\overline{P},\Delta}{\vdash [Q]\,;\,\Gamma,\Delta}$$

$$\frac{\vdash P\,;\,\Gamma}{\vdash\,;\,P,\Gamma} \qquad \frac{\vdash [Q]\,;\,\Gamma}{\vdash [Q]\,;\,\Gamma,A} \qquad \frac{\vdash [Q]\,;\,\Gamma,A,A}{\vdash [Q]\,;\,\Gamma,A}$$

$$\frac{\vdash P_1\,;\,\Gamma \quad \vdash P_2\,;\,\Delta}{\vdash P_1 \otimes P_2\,;\,\Gamma,\Delta} \qquad \frac{\vdash P_1\,;\,\Gamma}{\vdash P_1 \oplus P_2\,;\,\Gamma} \qquad \frac{\vdash P_2\,;\,\Gamma}{\vdash P_1 \oplus P_2\,;\,\Gamma} \qquad \frac{\vdash\,;\,N,\Gamma}{\vdash {\downarrow}N\,;\,\Gamma}$$

$$\frac{\vdash [Q]\,;\,N_1,N_2,\Gamma}{\vdash [Q]\,;\,N_1 \parr N_2,\Gamma} \qquad \frac{\vdash [Q]\,;\,N_1,\Gamma \quad \vdash [Q]\,;\,N_2,\Gamma}{\vdash [Q]\,;\,N_1 \,\&\, N_2,\Gamma} \qquad \frac{\vdash [Q]\,;\,P,\Gamma}{\vdash [Q]\,;\,{\uparrow}P,\Gamma}$$

Synthesizing Laurent's LLP

We apply the following two restrictions to weakly focalised classical logic :

– allow the weakening and contraction rules only for $A = N$,

– replace the two rules

$$\frac{\vdash P\,;\,\Gamma}{\vdash\,;\,P, \Gamma} \qquad \frac{\vdash [Q]\,;\,P, \Gamma}{\vdash [Q]\,;\,\uparrow P, \Gamma}$$

with the rule

$$\frac{\vdash P\,;\,\Gamma}{\vdash\,;\,\uparrow P, \Gamma}$$

Then it is then easy to check that in the resulting system the contexts (i.e. the formulas on the right of the stoup) are all negative (and hence the n–cut can never be applied)

Then the presence of a stoup is superfluous (there is no other positive formula that one might be tempted to change for !).

System LJ_0

One can then reorganise the sequents in “all positive form”, and the result is the following intutionnistic system (also called LJ_0).

Formulas : $P ::= X \parallel P{\otimes}P \parallel P{\oplus}P \parallel \neg^{+}P$ (where $\neg^{+}P$ stands for $\downarrow\overline{P}$)

$$\frac{}{P \vdash P} \qquad \frac{\mathcal{P}_1 \vdash P \quad \mathcal{P}_2, P \vdash [Q]}{\mathcal{P}_1, \mathcal{P}_2 \vdash [Q]}$$

$$\frac{\mathcal{P} \vdash [Q]}{\mathcal{P}, P \vdash [Q]} \qquad \frac{\mathcal{P}, P, P \vdash [Q]}{\mathcal{P}, P \vdash [Q]}$$

$$\frac{\mathcal{P}_1 \vdash P_1 \quad \mathcal{P}_2 \vdash P_2}{\mathcal{P}_1, \mathcal{P}_2 \vdash P_1 \otimes P_2} \qquad \frac{\mathcal{P} \vdash P_1}{\mathcal{P} \vdash P_1 \oplus P_2} \qquad \frac{\mathcal{P} \vdash P_2}{\mathcal{P} \vdash P_1 \oplus P_2} \qquad \frac{\mathcal{P}, P \vdash}{\mathcal{P} \vdash \neg^{+}P}$$

$$\frac{\mathcal{P}, P_1, P_2 \vdash [Q]}{\mathcal{P}, P_1 \otimes P_2 \vdash [Q]} \qquad \frac{\mathcal{P}, P_1 \vdash [Q] \quad \mathcal{P}, P_2 \vdash [Q]}{\mathcal{P}, P_1 \oplus P_2 \vdash [Q]} \qquad \frac{\mathcal{P} \vdash P}{\mathcal{P}, \neg^{+}P \vdash}$$

Translating weakly focalised classical logic to LJ_0

Weakly focalised classical proofs of $\left\{ \begin{array}{l} \vdash\, ;\, \mathcal{P}, \mathcal{N} \\ \vdash P\, ;\, \mathcal{P}, \mathcal{N} \end{array} \right\}$ translate straightforwardly to proofs of $\left\{ \begin{array}{l} \overline{\mathcal{N}}, {\downarrow}\overline{\mathcal{P}} \vdash \\ \overline{\mathcal{N}}, {\downarrow}\overline{\mathcal{P}} \vdash P \end{array} \right\}$ in LJ_0.

The reversing translation to linear logic

The key observation (that goes back to Quatrini-Tortora) is that the structural rules of linear logic apply "almost" to all negative formulas, thanks to their reversibility (whence the name reversing).

One defines the following translation on the formulas of LLP :

$$X^\rho = !X \quad (P \otimes Q)^\rho = P^\rho \otimes Q^\rho \quad (P \oplus Q)^\rho = P^\rho \oplus Q^\rho \quad (\neg^+ P)^\rho = !\overline{P^\rho}$$

Then the translation carries proofs of $\left\{ \begin{array}{l} \mathcal{P} \vdash \\ \mathcal{P} \vdash P \end{array} \right\}$ to linear logic proofs of $\left\{ \begin{array}{l} \vdash \overline{\mathcal{P}^\rho} \\ \vdash \overline{\mathcal{P}^\rho}, P^\rho \end{array} \right.$

But in fact the natural target of this translation is Tensor logic. And we learn this from model constructions.

Melliès' tensor logic TL

Formulas : $P ::= X \parallel P \otimes P \parallel P \oplus P \parallel \neg^{\circ} P \parallel !P$

$$\frac{}{P \vdash P} \qquad \frac{\mathcal{P}_1 \vdash P \quad \mathcal{P}_2, P \vdash [Q]}{\mathcal{P}_1, \mathcal{P}_2 \vdash [Q]}$$

$$\frac{\mathcal{P} \vdash [Q]}{\mathcal{P}, !P \vdash [Q]} \qquad \frac{\mathcal{P}, !P, !P \vdash [Q]}{\mathcal{P}, !P \vdash [Q]} \qquad \frac{\mathcal{P}, P \vdash [Q]}{\mathcal{P}, !P \vdash [Q]} \qquad \frac{!\mathcal{P} \vdash P}{!\mathcal{P} \vdash !P}$$

$$\frac{\mathcal{P}_1 \vdash P_1 \quad \mathcal{P}_2 \vdash P_2}{\mathcal{P}_1, \mathcal{P}_2 \vdash P_1 \otimes P_2} \qquad \frac{\mathcal{P} \vdash P_1}{\mathcal{P} \vdash P_1 \oplus P_2} \qquad \frac{\mathcal{P} \vdash P_2}{\mathcal{P} \vdash P_1 \oplus P_2} \qquad \frac{\mathcal{P}, P \vdash}{\mathcal{P} \vdash \neg^{\circ} P}$$

$$\frac{\mathcal{P}, P_1, P_2 \vdash [Q]}{\mathcal{P}, P_1 \otimes P_2 \vdash [Q]} \qquad \frac{\mathcal{P}, P_1 \vdash [Q] \quad \mathcal{P}, P_2 \vdash [Q]}{\mathcal{P}, P_1 \oplus P_2 \vdash [Q]} \qquad \frac{\mathcal{P} \vdash P}{\mathcal{P}, \neg^{\circ} P \vdash}$$

Categorical models

- focalised classical logic : control categories (Selinger)
- LJ_0 : response categories (i.e., cartesian and cocartesian categories where coproducts distribute over product, with an exponentiable objet R) (cf. e.g. Hofmann, Lafont, Reus, Streicher)
- We consider the following class of models for tensor logic (à la Lafont) : monoidal and cocartesian categories $\mathbf{C}$ where coproducts distribute over tensor, with a (linearly) exponentiable object R_L, and a right adjoint ! to the forgetful functor $U : \otimes{-}Com(\mathbf{C}) \to \mathbf{C}$.

Proposition : Given a model of TL à la Lafont, $\otimes{-}Com(\mathbf{C})$ (the category of comonoid objects of $\mathbf{C}$) is a response category

Proof of the proposition

- If $\mathbf{C}$ is symmetric monoidal, then $\otimes{-}Com(\mathbf{C})$ is cartesian (standard for any monoidal category $\mathbf{C}$.)
- If $\mathbf{C}$ further has distributive coproducts, then U creates distributive coproducts in $\otimes{-}Com(\mathbf{C})$: indeed, $P \oplus Q$ gets a comonoid structure by setting $\epsilon_{P\oplus Q} = [\epsilon_P, \epsilon_Q]$ and $\delta_{P\oplus Q} = \iota \circ (\delta_P \oplus \delta_Q)$, where ι is obtained by composing (in diagrammatic order) the injection into the coproduct that is isomorphic to $(P \oplus Q) \otimes (P \oplus Q)$, followed by that isomorphism.
- Finally, we set $R = !R_L$ and $R^P = !(P \multimap R_L)$, and we get

$$\begin{aligned}
\otimes{-}Com(\mathbf{C})[Q, R^P] &\cong \mathbf{C}[Q, P \multimap R_L] \\
&\cong \mathbf{C}[Q \otimes P, R_L] \\
&\cong \otimes{-}Com(\mathbf{C})[Q \otimes P, !R_L]
\end{aligned}$$

Generalising the reversing translation

$$X^\rho = !X \quad (P \otimes Q)^\rho = P^\rho \otimes Q^\rho \quad (P \oplus Q)^\rho = P^\rho \oplus Q^\rho \quad (\neg^+ P)^\rho = !\neg^\circ P^\rho$$

Then the translation carries proofs of $\left\{ \begin{array}{l} \mathcal{P} \vdash \\ \mathcal{P} \vdash P \end{array} \right\}$ to proofs of $\left\{ \begin{array}{l} \mathcal{P}^\rho \vdash \\ \mathcal{P}^\rho \vdash P^\rho \end{array} \right\}$ in TL.

Syntax of polarised CLC

A polarised version of CLC is needed to get the commutation advocated by Laurent and Regnier precisely right.

$$N ::= M \to N \parallel \neg P \qquad P ::= X \parallel \neg_0 N$$

We set $A ::= N \parallel P$.

$$\begin{array}{l} c ::= t^+ t^- \parallel t^- t^+ \\ t^+ ::= x^+ \parallel \lambda x^-.c \\ t^- ::= l^- \parallel x^- \parallel \lambda x^+.c \parallel \lambda x^-.t^- \parallel t^- t^- \\ x ::= x^+ \parallel x^- \end{array}$$

In the typing rules, contexts have the form $\Gamma; [l^- : N]$, where Γ is a list of declarations $x : A$, and $[l^- : N]$ is either empty or a single (linear) declaration.

Typing rules of polarised CLC

$$\frac{}{\Gamma, x : A ; \vdash x : A} \qquad \frac{}{\Gamma ; l^- : N \vdash l^- : N}$$

$$\frac{\Gamma, x^- : N_1 ; [M] \vdash t^- : N_2}{\Gamma ; [M] \vdash \lambda x^-.t^- : N_1 \to N_2} \qquad \frac{\Gamma ; [M] \vdash t_1^- : N_1 \to N_2 \qquad \Gamma ; \vdash t_2^- : N_1}{\Gamma ; [M] \vdash t_1^- t_2^- : N_2}$$

$$\frac{c : (\Gamma ; l^- : N \vdash)}{\Gamma ; \vdash \lambda l^-.c : \neg_0 N} \qquad \frac{\Gamma ; \vdash t_1^+ : \neg_0 N \qquad \Gamma ; [M] \vdash t_2^- : N}{t_1^+ t_2^- : (\Gamma ; [M] \vdash)}$$

$$\frac{c : (\Gamma, x^+ : P ; [M] \vdash)}{\Gamma ; [M] \vdash \lambda x^+.c : \neg P} \qquad \frac{\Gamma ; [M] \vdash t_1^- : \neg P \qquad \Gamma ; \vdash t_2^+ : P}{t_1^- t_2^+ : (\Gamma ; [M] \vdash)}$$

We note that the judgements are of either of the two forms

$$\Gamma ; [M] \vdash [N] \qquad \Gamma ; \vdash P$$

Translation of CBN $\lambda\mu$ calculus into CLC

One translates formulas and judgements as follows :

$$(\overline{X})^K = \neg X \qquad (M \to N)^K = M^K \to N^K$$

$$\text{if } \left\{ \begin{array}{l} c : (\mathcal{N}_1 \vdash \mathcal{N}_2) \\ \mathcal{N}_1 \vdash v : N \,|\, \mathcal{N}_2 \end{array} \right\}, \text{ then } \left\{ \begin{array}{l} c^K : (\mathcal{N}_1^K, \neg_0 \mathcal{N}_2^K \,;\, \vdash) \\ \mathcal{N}_1^K, \neg_0 \mathcal{N}_2^K \,;\, \vdash v^K : N^K \end{array} \right.$$

where more precisely, each $x : N$ in $\mathcal{N}_1$ becomes $x : N^K$, and each $\alpha : N$ in $\mathcal{N}_2$ becomes $x_\alpha^+ : \neg_0 N^K$.

$$x^K = x \qquad (\lambda x.v)^K = \lambda x.v^K \qquad (v_1 v_2)^K = v_1^K v_2^K \qquad ([\alpha]v)^K = x_\alpha^+ v^K$$

where, for a negative formula N of CLC, $C_N : \neg\neg_0 N \to N$ is defined as follows by induction :

$$C_{\neg P} = \lambda x^-.\lambda y^+.x^-(\lambda l^-.l^- y^+)$$
$$C_{M \to N} = \lambda x^-.\lambda y^-.C_N(\lambda z^+.x^-(\lambda f^-.z^+(f^- y^-)))$$

Translation of CLC to TL

$$X^\gamma = X \quad (\neg_0 N)^\gamma = N_\gamma$$
$$(M \to N)_\gamma = (!\neg^\circ M_\gamma) \otimes N_\gamma \quad (\neg P)_\gamma = !P^\gamma$$
$$\gamma(\Gamma, X) = \gamma(\Gamma), !X \quad \gamma(\Gamma, \neg_0 N) = \gamma(\Gamma), N_\gamma \quad \gamma(\Gamma, N) = !\neg^\circ N_\gamma$$

$$\text{if } \left\{ \begin{array}{l} \Gamma \,;\, [M] \vdash [N] \\ \Gamma \,;\, \vdash P \end{array} \right\}, \text{ then } \left\{ \begin{array}{l} \gamma(\Gamma), [N_\gamma] \vdash [M_\gamma] \\ \gamma(\Gamma) \vdash P^\gamma \end{array} \right.$$