Bi-Intuitionism as dialogue chirality

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0. Plan of the talk.

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1. C. Rauszer's Bi-intuitionism.

• **Heyting algebra**: a bounded lattice $\mathcal{A} = (A, \vee, \wedge, 0, 1)$ with *Heyting implication* $(\rightarrow)$, defined as the right adjoint to meet. Thus
• **co-Heyting algebra** is a lattice $\mathcal{C}$ such that $\mathcal{C}^{op}$ is a Heyting algebra.
$\mathcal{C} = (C, \vee, \wedge, 1, 0)$ with *subtraction* $(\smallsetminus)$ defined as the left adjoint of join.

Heyting algebra	co-Heyting algebra
$\dfrac{c \wedge b \ \leq \ a}{c \ \leq \ b \rightarrow a}$	$\dfrac{a \ \leq \ b \vee c}{a \smallsetminus b \ \leq \ c}$

• **Bi-Heyting** algebra: a lattice with the structure of Heyting and of co-Heyting algebra.

1.1. Rauszer's Bi-Intuitionistic logic.

- **Bi-intuitionistic language:**

$A, B \ := \ a \mid \top \mid \bot \mid A \wedge B \mid A \to B \mid A \vee B \mid A \smallsetminus B$

Read $A \smallsetminus B$ as *"A excludes B"*.

- **Kripke models** [Rauszer 1977]:

$(W, \leq, \Vdash)$, with $(W, \leq)$ a preorder;

- $w \Vdash A \to B$ iff $\forall w' \geq w. w' \Vdash A$ implies $w' \Vdash B$;

- $w \Vdash A \smallsetminus B$ iff $\exists w' \leq w. w' \Vdash A$ and not $w' \Vdash B$.

Gödel, McKinsey and Tarsky translation in **tensed S4:**

- implication must hold in all *future* world;
- subtraction must hold in some *past* world.
- *monotonicity* holds for all formulas.

$(A \to B)^M = \Box(A^M \to B^M)$ (*necessity in the future*)

$(A \smallsetminus B)^M = \diamondminus(A^M \wedge \neg B^M)$ (*possibility in the past*)

• **Strong negation:** $\sim A =_{df} A \to \bot \quad (\sim A)^M = \Box\neg A$.
• **Weak negation:** $\frown A =_{df} \top \setminus A \quad (\frown A)^M = \diamondminus\neg A$.

Notation: We reserve '$\neg A$' for *classical negation*.
Write $(\sim\frown)^{n+1}A = \sim\frown (\sim\frown)^n A$, $(\sim\frown)^0 A = A$
and similarly $(\frown\sim)^n A$.

Fact:
$(\sim\frown)^{n+1}A \Rightarrow (\sim\frown)^n A$ *but not conversely*, for all $n \geq 0$.
$(\frown\sim)^n A \Rightarrow (\frown\sim)^{n+1} A$ *but not conversely*, for all $n \geq 0$.

• How to formalize Bi-intuitionism in a Gentzen system?

$$\to\text{-R}\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B, \Delta}(*) \qquad \to\text{-L}\ \frac{\Gamma_1 \Rightarrow \Delta_1 A \quad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, A \to B, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

$$\setminus\text{-R}\ \frac{\Gamma_1 \Rightarrow \Delta_1, C \quad D, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, C \setminus D, \Delta_2} \qquad \setminus\text{-L}\ \frac{C \vdash D, \Delta}{\Gamma, C \setminus D \Rightarrow \Delta}(**)$$

Cut-elimination fails: (T. Uustalu)

$(q \vee p) \setminus q \Rightarrow r \to (p \wedge r)$ is provable with cut from

$(q \vee p) \setminus q \Rightarrow p$ and $p \Rightarrow r \to (p \wedge r)$, but there is no

cut-free proofs satisfying conditions $(*)$ and $(**)$.

Intuitionistic formalization is non trivial (see [Crolard 2001, 2004] [Pinto & Uustalu 2010]).

2. No categorical model for Rauszer's logic.

Joyal's Theorem. *Let $\mathcal{C}$ be a CCC with an initial object $\bot$. Then for any object A in $\mathcal{C}$, if $\mathcal{C}(A, \bot)$ is nonempty, then A is initial.*

Proof: $\bot \times A$ is initial, as $\mathcal{C}((\bot \times A), B) \approx \mathcal{C}(\bot, B^A)$. Given $f : A \to \bot$, show that $A \approx \bot \times A$, using the fact that $\langle f, id_A \rangle \circ \pi'_{\bot,A} = id_{\bot,A}$, since $\bot \times A$ is initial.

Crolard's Theorem. *If both $\mathcal{C}$ and $\mathcal{C}^{op}$ are CCCs, then $\mathcal{C}$ is a preorder.*

Proof: Let $A \oplus B$ be the *coproduct* and A_B the *co-exponent* of A and B.
Then $\mathcal{C}(A, B) \approx \mathcal{C}(A, \bot \oplus B) \approx \mathcal{C}(A_B, \bot)$. By Joyal's Theorem $\mathcal{C}(A_B, \bot)$ contains at most one arrow.

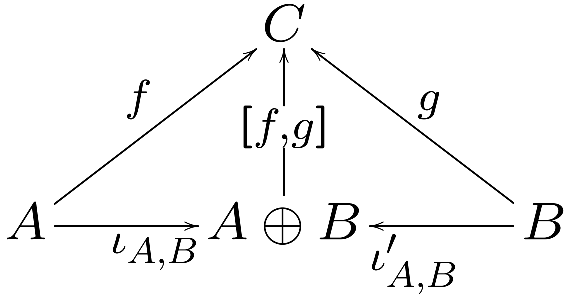
2.1. No problem in the linear case:

Multiplicative linear Intuitionistic: $\mathcal{A} = (A, \mathbf{1}, \otimes, \multimap)$ [with natural iso's], *symmetric monoidal closed* (with $\multimap$ the right adjoint of $\otimes$).
Multiplicative linear co-Intuitionistic: $\mathcal{C} = (C, \bot, \wp, \smallsetminus)$ [with natural iso's], *symmetric monoidal left-closed* (with $\smallsetminus$ the left adjoint of $\wp$).

No problem in combining two structures, one monoidal closed, the other monoidal left-closed.

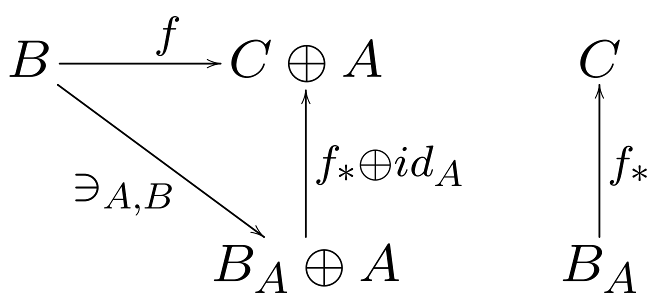
• No modelling of **co-Intuitionism** in **Set** since *disjunction* (**coproduct**) is *disjoint union*.

Recall: The coproduct of A and B is an object $A \oplus B$ together with arrows $\iota_{A,B}$ and $\iota'_{A,B}$ such that for every C and every pair of arrows $f : A \to C$ and $g : B \to C$ there is a unique $[f, g] : A \oplus B \to C$ making the following diagram commute:



3. No model of Co-Intuitionism in Set.

Recall: The *co-exponent* of A and B is an object B_A together with an arrow $\ni_{A,B}$: $B \to B_A \oplus A$ such that for any arrow $f : B \to C \oplus B$ there exists a unique $f_* : B_A \to C$ making the following diagram commute:



Crolard's Lemma: *The co-exponent B_A of two sets A and B is defined iff $A = \emptyset$ or $B = \emptyset$.*

Proof: In **Set** the *coproduct* is the disjoint union and the initial object is $\emptyset$.
(if) For any B, let $B_\perp =_{df} B$ with $\ni_{\perp,B} =_{df} \iota_{B,\perp}$.
For any A, let $\perp_A =_{df} \perp$ with $\ni_{A,\perp} =_{df} \Box : \perp \to \perp \oplus A$.
(only if) If $A \neq \emptyset \neq B$ then the functions f and $\ni_{A,B}$ for every $b \in B$ must *choose a side*, left or right, of the coproduct in their target and moreover $f_\star \oplus id_A$ leaves the side unchanged. Hence, if we take a nonempty set C and f with the property that for some b different sides are chosen by f and $\ni_{A,B}$, then the diagram does not commute.

4. Dialogue chirality.

A dialogue chirality on the left is a pair of monoidal categories $(\mathcal{A}, \wedge, \text{true})$ and $(\mathcal{B}, \vee, \text{false})$ equipped with an adjunction

$$\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{B} \quad (L \dashv R)$$

whose unit and counit are denoted as

$$\eta : Id \quad \rightarrow \quad R \circ L \qquad \epsilon : L \circ R \rightarrow Id$$

together with a monoidal functor

$$(-)^* \quad ; \quad \mathcal{A} \quad \rightarrow \quad \mathcal{B}^{op}$$

and a family of bijections

$$\chi_{m,a,b} \quad : \quad \langle m \wedge a | b \rangle \quad \rightarrow \quad \langle a | m^* \vee b \rangle$$

natural in m, a, b *(curryfication)*. Here the bracket $\langle a|b \rangle$ denotes the set of morphisms from a to $R(b)$ in the category $\mathcal{A}$:

$$\langle a|b \rangle \quad = \quad \mathcal{A}(a, R(b)).$$

The family χ is moreover required to make the diagram

$$\begin{array}{ccccc}
\langle (m \wedge n) \wedge a \mid b \rangle & & \xrightarrow{\quad \chi_{m \wedge n} \quad} & & \langle a \mid (m \wedge n)^* \vee b \rangle \\
\Big\downarrow \scriptstyle{assoc.} & & = & & \Big\uparrow \scriptstyle{assoc.\ monoid.\ of\ (-)^*} \\
\langle m \wedge (n \wedge a) \mid b \rangle & \xrightarrow{\chi_m} & \langle n \wedge a \mid m^* \vee b \rangle & \xrightarrow{\chi_n} & \langle a \mid n^* \vee (m^* \vee b) \rangle
\end{array}$$

commute for all objects a, m, n, and all morphisms $f : m \to n$ of the category $\mathcal{A}$ and all objects b of the category $\mathcal{B}$.

4.1. Modelling Bi-intuitionism.

- Let $\mathcal{A}$ be a model of **Int conjunctive logic** on the language $\cap$, $\top$. ($\mathcal{A}$ may be Cartesian).
- $\mathcal{B}$ a model of **co-Int disjunctive logic** on the language $\curlyvee$, $\bot$.

Give a suitable sequent-calculus formalization of **Int** and **co-Int** and work with the free categories built from the syntax.

4.1. Modelling Bi-intuitionism (cont).

- The contravariant monoidal functor $(\ )^* : \mathcal{A} \to \mathcal{B}^{op}$ models "De Morgan duality":

$$(A_1 \cap A_2)^* \;=\; A_1^* \curlyvee A_2^*$$

- There is a dual contravariant functor $^*(\ ) : \mathcal{B} \to \mathcal{A}^{op}$.

$$^*(C_1 \curlyvee C_2) \;=\; {}^*C_1 \cap {}^* C_2$$

- **What are the covariant functors** $L \dashv R$?
- **Main Idea:**

introduce negations $\sim: \mathcal{A} \to \mathcal{A}$ and $\frown: \mathcal{B} \to \mathcal{B}$.

[In the chirality model $\sim A$ and $\frown C$ may be primitive.]

- Let $\mathbf{u}$ be a specified object of $\mathcal{A}$

- Think of $\sim A =_{df} A \supset \mathbf{u}$ (notation: $\sim_{\mathbf{u}} A$).

- Let $\mathbf{j}$ be a specified object of $\mathcal{B}$

- Think of $\frown C =_{df} \mathbf{j} \smallsetminus C$ (notation: ${}_{\mathbf{j}} \frown C$).

- *Let* $L =_{df} \frown ({}^*(\_))$ *and* $R =_{df} \sim ((\_)^*)$.

5. Polarized Bi-Intuitionism.

Language of polarized bi-intuitionism $\mathbf{BI}_p$:
- sets of elementary formulas $\{a_1, \ldots\}$ and $\{c_1, \ldots\}$;

$$A, B \; := a \mid \top \mid \mathbf{u} \mid A \cap B \mid \sim A \mid A \supset B \mid \; C^{\perp}$$
$$C, D \; := c \mid \bot \mid \mathbf{j} \mid C \curlyvee D \mid \; \frown C \mid C \setminus D \mid A^{\perp}$$

5.1. Informal intended interpretation.

Logic for pragmatics: *an intensional 'justification logic' of* **assertions** *and* **hypotheses**.

• **Propositional letters** $p_1, \ldots$ (countably many);
• $\vdash\_$ and $\mathcal{H}\_$ are *illocutionary force* operators for *assertion* and *hypothesis* (Austin).
Elementary formulas: $a_i = \vdash p_i, \quad c_i = \mathcal{H} p_i.$

What justifies an assertion / a hypothesis?
• Only **"conclusive evidence"** justifies assertions,
• a **"scintilla of evidence"** justifies hypotheses.

5.2. A BHK interpretation of the logic of assertions and hypotheses.

- $a_i = \vdash p_i$ the type of *evidence for assertions* of p_i;
- $c_j = \mathcal{H} p_j$ the type of *evidence for hypotheses* that p_j;
- $A \supset B =$ the type of methods transforming assertive evidence for A into assertive evidence for B;
- $C \smallsetminus D$ ("C excludes D") = the type of hypothetical evidence that C is justified and D cannot be justified;
- $\mathbf{u} =$ an assertion *always unjustified*;
- $\mathbf{j} =$ a hypothesis *always justified*;
- $\sim A, C^{\perp} =$ *denial* of A, C;
- $\frown C, A^{\perp} =$ *doubt* about C, A.

Questions: (i) What is a *scintilla of evidence*? a *doubt about* an assertion or a hypothesis?

Comment: *Scintilla of evidence* is legal terminology [Gordon & Walton 2009]. It evokes probabilistic methods, perhaps infinitely-valued logics.

An alternative: define **evidence for** and **evidence against** assertion and hypotheses. Obtain a "Dialectica-like" dialogue semantics [Bellin *et al* 2014].

6. McKinsey-Tarski-Gödel's S4 translation

- Translation in *non-tensed* **S4**.
- *Monotonicity* holds for assertive formulas.
- *Anti-monotonicity* holds for hypothetical formulas.

$$
\begin{array}{rcl@{\quad}rcl}
(\vdash p)^M & = & \Box p & (\mathcal{H} p)^M & = & \Diamond p, \\
(A \supset B)^M & = & \Box(A^M \to B^M) & (C \smallsetminus D)^M & = & \Diamond(C^M \wedge \neg D^M), \\
(\top)^M & = & \mathbf{t}, & (\bot)^M & = & \mathbf{f} \\
(A \cap B)^M & = & A^M \wedge B^M & (C \curlyvee D)^M & = & C^M \vee D^M, \\
(\sim A)^M & = & \Box\neg A^M & (\frown X)^M & = & \Diamond\neg X^M \\
(C^\perp)^M & = & \neg C^M & (A^\perp)^M & = & \neg A^M
\end{array}
$$

Lemma: $A^M \equiv \Box A^M$, $C^M \equiv \Diamond C$.

Note: $(\sim A)^M = \Box\neg\Box A^M = \Box\Diamond\neg A^M$, $\quad (C^\perp)^M = \neg\Diamond C^M = \Box\neg C^M$; symmetrically for $(\frown C)^M$ and $(A^\perp)^M$.
Negations and dualities are translated differently.
Note: $(C \smallsetminus D)^M = \Diamond(C^M \wedge \Box\neg D^M)$.

Some Facts.

- $(A^{\perp\perp})^M = \neg\neg A^M = A^M$; $(C^{\perp\perp})^M = \neg\neg C^M = C^M$.

- $(\sim\sim A)^M = \Box\neg\Box\neg A^M = \Box\Diamond A^M$;
- $(\frown\frown C)^M = \Diamond\neg\Diamond\neg C^M = \Diamond\Box C^M$.

- $(\sim\frown A)^M = \Box\neg\Diamond\neg A^M = \Box\Box A^M = A^M$
- $(\frown\sim C)^M = \Diamond\neg\Box\neg C^M = \Diamond\Diamond C^M = C^M$

Thus $(\sim\frown)^n A \Leftrightarrow A$, $(\frown\sim)^n C \Leftrightarrow C$, for all n.

- $(\sim\frown C)^M = \Box\neg\Diamond\neg C^M = \Box C^M = (\sim (C^\perp))^M$
- $(\frown\sim A)^M = \Diamond\neg\Box\neg A^M = \Diamond A^M = (\frown (A)^\perp)^M$.

Thus $(\sim\frown)^n C \Leftrightarrow \sim\frown C$, $(\frown\sim)^n A \Leftrightarrow \frown\sim A$, for all $n \geq 1$.

Expectation ($\mathcal{E}p$) and Conjecture ($\mathcal{C}p$).

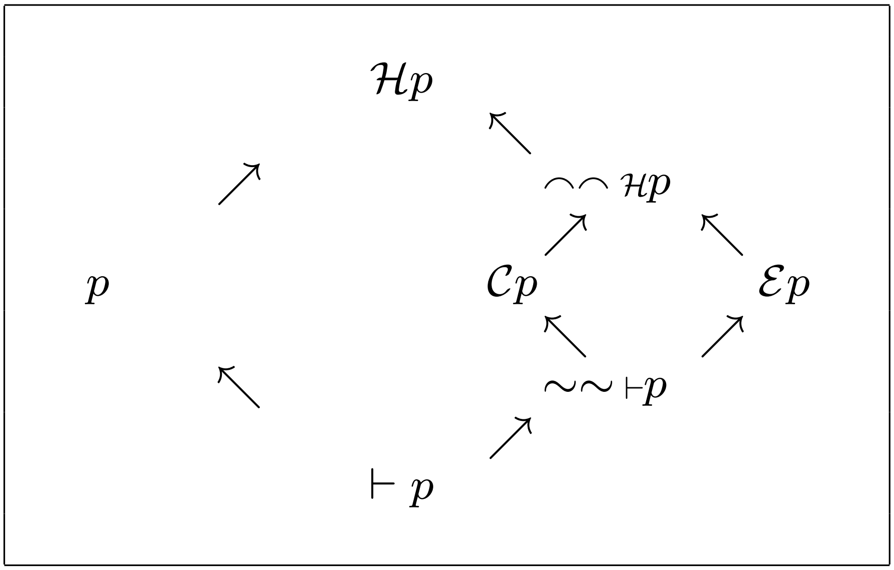
Idea: $\mathcal{E}p = \sim ((\mathcal{H}p)^\perp)$. *Expecting that p* is *denying the denial of the hypothesis p*, i.e., *asserting that in all situations the hypothesis p would be justified.*

$\mathcal{C}p = \frown ((\vdash p)^\perp)$. *Conjecturing that p* is *doubting that there may be doubts about the assertion of p*, i.e., *making the hypothesis that in some situation p may be assertable.*

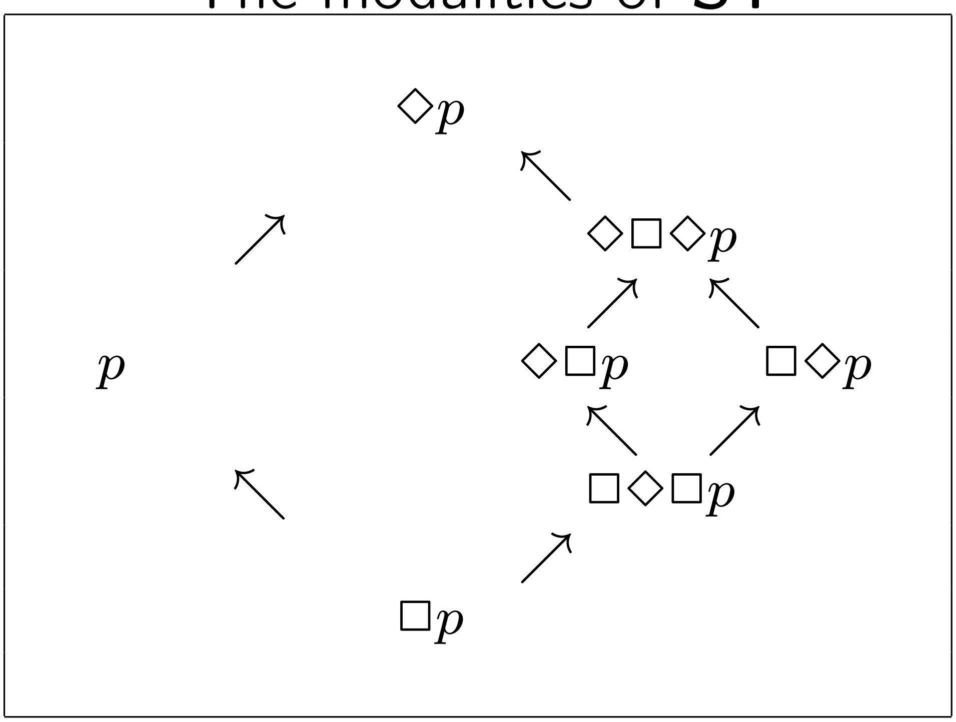
Notice that $\mathcal{E}p = R\ \mathcal{H}p$ and $\mathcal{C}p = L \vdash p$.

6.1. Expectations ($\mathcal{E}p$), conjectures ($\mathcal{C}p$).

Assertions, hypotheses, conjectures, expectations



The modalities of **S4**



7. Bi-polar sequent calculus BI_p.

$$\Gamma\ ;\ \Rightarrow\ A\ ;\ \Delta \qquad \text{or} \qquad \Gamma\ ;\ C \Rightarrow\ ;\ \Delta$$

int: $\Delta^{\perp}, \Gamma\ ; \Rightarrow\ A;$ **co-int:** $;\ C \Rightarrow\ ;\ \Delta, \Gamma^{\perp}$

Write $\Gamma\ ;\ \epsilon \Rightarrow \epsilon'\ ;\ \Delta$, with exactly one of ϵ, ϵ' non-null.

Identity Rules:

logical axiom: $A\ ;\ \Rightarrow\ A\ ;$

logical axiom: $;\ C\ \Rightarrow\ ;\ C$

cut$_1$:

$$\frac{\Theta\ ;\ \Rightarrow\ A\ ;\ \Upsilon \qquad A, \Theta'\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon'}{\Theta, \Theta'\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \Upsilon'}$$

cut$_2$:

$$\frac{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, C \qquad \Theta'\ ;\ C\ \Rightarrow\ \Upsilon'}{\Theta, \Theta'\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \Upsilon'}$$

Proper axioms of the pragmatic interpretation

$$\vdash p\ ;\ \mathbf{j}\ \Rightarrow\ ;\ \mathcal{H}p \qquad \vdash p\ ;\ \Rightarrow\ \mathbf{u}\ ;\ \mathcal{H}p$$

Duality Rules:

$\bot_{ci}$ right:

$$\frac{\Theta \; ; \; C \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; C^{\bot} \; ; \; \Upsilon}$$

$\bot_{ci}$ left:

$$\frac{\Theta ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, C}{C^{\bot}, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

$\bot_{ic}$ right:

$$\frac{\Theta, A \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, A^{\bot}}$$

$\bot_{ic}$ left:

$$\frac{\Theta ; \; \Rightarrow \; A \; ; \; \Upsilon}{\Theta \; ; \; A^{\bot} \; \Rightarrow \; ; \; \Upsilon}$$

$\mathbf{u/j}$ *left*

$$\mathbf{u} \; ; \; \mathbf{j} \; \Rightarrow \; ;$$

$\mathbf{u/j}$ *right*

$$; \; \Rightarrow \; \mathbf{u} \; ; \; \mathbf{j}$$

Structural Rules:

contraction left

$$\frac{A, A, \Theta \; ; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{A, \Theta \; ; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

contraction right

$$\frac{\Theta \; ; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, C.C}{\Theta \; ; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, C}$$

weakening left

$$\frac{\Theta \; ; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{A, \Theta \; ; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

weakening right

$$\frac{\Theta \; ; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\Theta \; ; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, C}$$

Conjunction and Disjunction

assertive validity axiom:

$$\Theta \;;\; \Rightarrow \; \top \;;\; \Upsilon$$

$\cap$ *right:*

$$\frac{\Theta \;;\; \Rightarrow \; A_1 \;;\; \Upsilon \qquad \Theta \;;\; \Rightarrow \; A_2 \;;\; \Upsilon}{\Theta \;;\; \Rightarrow \; A_1 \cap A_2 \;;\; \Upsilon}$$

$\cap$ *left:*

$$\frac{A_i, \Theta \;;\; \epsilon \; \Rightarrow \; \epsilon' \;;\; \Upsilon}{A_0 \cap A_1, \Theta \;;\; \epsilon \; \Rightarrow \; \epsilon' \;;\; \Upsilon}$$

for $i = 0, 1$.

hypothetical absurdity axiom:

$$\Theta \;;\; \bot \; \Rightarrow \;;\; \Upsilon$$

$\curlyvee$ right:

$$\frac{\Theta \;;\; \epsilon \; \Rightarrow \; \epsilon' \;;\; \Upsilon, C_0, C_1}{\Theta \;;\; \epsilon \; \Rightarrow \; \epsilon' \;;\; \Upsilon, C_0 \curlyvee C_1}$$

$\curlyvee$ left:

$$\frac{\Theta_1 \;;\; C_1 \; \Rightarrow \;;\; \Upsilon_1 \qquad \Theta_2 \;;\; C_2 \; \Rightarrow \;;\; \Upsilon_2}{\Theta_1, \Theta_2 \;;\; C_1 \curlyvee C_2 \; \Rightarrow \;;\; \Upsilon_1, \Upsilon_2}$$

Implication and Subtraction

$\supset$ *right:*

$$\frac{\Theta, A_1 \ ; \ \Rightarrow \ A_2 \ ; \ \Upsilon}{\Theta \ ; \ \Rightarrow \ A_1 \supset A_2 \ ; \ \Upsilon}$$

$\supset$ *left :*

$$\frac{\Theta_1; \ \Rightarrow \ A_1 \ ; \ \Upsilon_1 \qquad A_2, \Theta_2 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon_2}{A_1 \supset A_2, \Theta_1, \Theta_2 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon_1, \Upsilon_2}$$

$\diagdown$ *right:*

$$\frac{\Theta_1 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon_1, C_1 \qquad \Theta_2 \ ; \ C_2 \ \Rightarrow \ ; \ \Upsilon_2}{\Theta_1, \Theta_2 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon_1, \Upsilon_2, C_1 \diagdown C_2}$$

$\diagdown$ *left:*

$$\frac{\Theta; \ C_1 \ \Rightarrow \ ; \ \Upsilon, C_2}{\Theta \ ; \ C_1 \diagdown C_2 \ \Rightarrow \ ; \ \Upsilon}$$

8. Categorical model of BI_p

We show that categorical models of $\mathbf{BI}_p$ have the form of dialogue chirality.
We sketch the construction of the syntactic category:
- **objects** are formulas;
- **morphisms** are equivalence classes of sequent derivations;
- subject to naturality conditions [omitted].

• Let $\mathcal{A} = (\mathbf{Int}, \cap, \top)$ the cartesian category of intuitionistic fomulas and derivations in $\mathbf{BI}_p$.
• Let $\mathcal{B} = (\mathbf{co-Int}, \curlyvee, \perp)$ the monoidal category of co-intuitionistic formulas and derivations in $\mathbf{BI}_p$.

• We have operations $\sim: \mathcal{A} \to \mathcal{A}$ (written $\sim_u$) and $\frown: \mathcal{B} \to \mathcal{B}$ (written $_j \frown$).
Let $\diamond(A) =_j\frown (A^\perp)$ and $\boxdot(C) =\sim_u (C^\perp)$.

• Define a functor $L = \diamond : \mathcal{A} \to \mathcal{B}$ sending a derivation $d \; : \; A_1; \Rightarrow A_2;$ to the derivation $\diamond d \; : \; ; \diamond A_1 \Rightarrow ; \diamond A_2$ defined in the obvious way.
Similarly define a functor $R = \boxdot : \mathcal{B} \to \mathcal{A}$.

• $L \dashv R$: the unit and co-unit of the adjunction are given by the derivations of Proposition (ii).

• The duality $(\_)^\perp$ is a contravariant monoidal functor $\mathcal{A} \to \mathcal{B}^{op}$, sending $d \; : \; A_1 \cap A_2; \Rightarrow A_3 \cap A_4;$ to $d^\perp \; : \; ; A_3^\perp \curlyvee A_4^\perp \Rightarrow ; A_1^\perp \curlyvee A_2^\perp ;$.

• Let $\langle A|C \rangle$ be the set of (equivalence classes of) sequent derivations of $A; \Rightarrow \boxdot C;$.

• $\mathcal{A} = (\mathcal{A}, \cap, \supset, \top)$ is in fact cartesian closed, so there is a natural bijection between $\mathcal{A}'(M \cap A, \boxdot\, C)$ and $\mathcal{A}'(A, M \supset \boxdot C)$.

• The provable equivalences of Proposition (iii) provide a natural bijection between $\mathcal{A}'(A, M \supset \boxdot\, C)$ and $\mathcal{A}'(A, \boxdot(M^{\perp} \curlyvee C))$ (*"De Morgan definition" of* $\supset$).

• By composing, we obtain the family of natural bijections

$$\chi_{M,A,C} : \langle M \cap A | C \rangle \rightarrow \langle A | M^{\perp} \curlyvee C \rangle.$$

Proposition: The following are provable in $\mathbf{BI}_p$.

(i) $\sim (A^{\perp}) \iff A$ and dually, $C \iff \frown (C^{\perp})$.

(ii) A ; $\Rightarrow \boxdot \Diamond A$; and ; $\Diamond \boxdot C \Rightarrow$; C.

(iii) $M \supset \boxdot C \iff \boxdot((M^{\perp}) \curlyvee C)$.

Proof. (ii) and (iii)

$$\perp_{ci}L\,\dfrac{\diagdown R\,\dfrac{; \Rightarrow \mathbf{u} ; \mathbf{j} \qquad \perp_{ci}L\,\dfrac{A ; \Rightarrow A ;}{A ; A^{\perp} \Rightarrow ;}}{A ; \Rightarrow \mathbf{u} ; \underbrace{{}_{j}\frown (A^{\perp})}_{\diamonddot A}}}{\supset R\,\dfrac{A, (\diamonddot A)^{\perp} ; \Rightarrow \mathbf{u} ;}{A ; \Rightarrow \underbrace{\sim_{u} ((\diamonddot A)^{\perp})}_{\boxdot \diamonddot A};}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{; C \Rightarrow ; C}{\Rightarrow C^{\perp} ; C}\,\perp_{ci}R \qquad \mathbf{u} ; \mathbf{j} \Rightarrow ;}{\underbrace{\sim_{u} (C^{\perp})}_{\boxdot C} ; \mathbf{j} \Rightarrow ; C}\,\supset L}{; \mathbf{j} \Rightarrow ; (\boxdot C)^{\perp}, C}\,\sim R}{\underbrace{{}_{j}\frown ((\boxdot C)^{\perp}}_{\diamonddot \boxdot C} ; \Rightarrow ; C}\,\diagdown L$$

$$\supset R\,\dfrac{\perp_{ci}L\,\dfrac{\curlyvee R\,\dfrac{\perp_{ic}R\,\dfrac{\supset L\,\dfrac{M ; \Rightarrow M ; \qquad \supset L\,\dfrac{\perp_{ci}R\,\dfrac{; C \Rightarrow ; C}{; \Rightarrow C^{\perp} ; C} \qquad \mathbf{u} ; \Rightarrow \mathbf{u} ;}{\boxdot C ; \Rightarrow \mathbf{u} ; C}}{M, M \supset \boxdot C ; \Rightarrow \mathbf{u} ; C}}{M \supset \boxdot C ; \Rightarrow \mathbf{u} ; M^{\perp}, C}}{M \supset \boxdot C ; \Rightarrow \mathbf{u} ; M^{\perp} \curlyvee C}}{M \supset \boxdot C, (\frown M^{\perp} \curlyvee C)^{\perp} ; \Rightarrow \mathbf{u} ;}}{M \supset \boxdot C ; \Rightarrow \boxdot (M^{\perp} \curlyvee C) ;}$$

$$\supset R\,\dfrac{\supset R\,\dfrac{\perp_{ci}L\,\dfrac{\supset L\,\dfrac{\perp_{ci}R\,\dfrac{\curlyvee L\,\dfrac{\dfrac{M ; \Rightarrow M ;}{M ; M^{\perp} \Rightarrow ;}\,\perp_{ic}L \qquad ; C \Rightarrow ; C}{M ; M^{\perp} \curlyvee C \Rightarrow ; C}}{M ; \Rightarrow M^{\perp} \curlyvee C ; C} \qquad \mathbf{u} ; \Rightarrow \mathbf{u} ;}{\sim_{u} (M^{\perp} \curlyvee C)^{\perp}, M ; \Rightarrow \mathbf{u} ; C}}{\boxdot (M^{\perp} \curlyvee C), M, C^{\perp} ; \Rightarrow \mathbf{u} ;}}{\boxdot (M^{\perp} \curlyvee C), M ; \Rightarrow \boxdot C ;}}{\boxdot ((\frown M) \curlyvee C) ; \Rightarrow M \supset \boxdot C ;}$$

9. An inductive classical type and $\lambda\mu$.

- **Type of 'expectations':** the collection of formulas $\mathcal{E}p_i$ (also written $\boxdot c_i$, for $c_i = \mathcal{H}p_i$).
- **Constructor** of the type of expectations: the operation $\boxdot(\_) =\sim_u ((\_)^\perp) : \mathbf{co}{-}\mathbf{Int} \to \mathbf{Int}$, corresponding to the covariant functor $R : \mathcal{B} \to \mathcal{A}$ of the chirality. This has a familiar name:

$$\mu \frac{\overline{x} : \Gamma \ ; \ \vdash t : \mathbf{u} \ ; \ \alpha : \mathcal{H}p_i, \ \overline{\alpha} : \Delta}{\overline{x} : \Gamma \ ; \ \vdash \ \mu\alpha.t : \underbrace{\boxdot\mathcal{H}p_i}_{\mathcal{E}p_i} \ ; \ \overline{\alpha} : \Delta} \mathcal{E} \text{ intro}$$

$$[\alpha] \frac{\overline{x} : \Gamma \vdash \ t : \overbrace{\boxdot\mathcal{H}p_i}^{\mathcal{E}p_i}; \ \overline{\alpha} : \Delta \qquad ; \alpha : \mathcal{H}p_i \vdash ; \alpha : \mathcal{H}p_i}{\overline{x} : \Gamma \ ; \ \vdash [\alpha]t : \mathbf{u} \ ; \ \alpha : \mathcal{H}p_i, \overline{\alpha} : \Delta} \mathcal{E} \text{ elim}$$

$\alpha : c_i$ possibly occurring in $\overline{\alpha} : \Delta$.

Clearly $\sim_u\sim_u\boxdot c \vdash \boxdot c$, since $\boxdot c =\sim_u (c^\perp)$.

Since $\boxdot\,\mathcal{H}p \neq \vdash p$ the classical *expectation type* lives within intuitionistic logic.

The same holds for the type of conjectures, defined as $\mathcal{C}p =_{df} \diamond(\vdash p) =_j \frown ((\vdash p)^\perp)$. Here we have $\diamond a \vdash_j \frown_j \frown \diamond a$, for $a = \vdash p$.

9.1. The $\lambda\mu$ calculus.

The untyped case: we are given
- a countable sequence of *variables* $x_1, x_2, \ldots$;
- a countable sequence of *names* $\alpha_1, \alpha_2, \ldots$.

$$\textbf{Terms}: \qquad t \ := \ x \mid \alpha \mid \lambda x.t \mid (t_1 t_2) \mid \mu\alpha.t \mid [\alpha]t$$

Reductions:

(β)	$(\lambda x.u)v$	$\triangleright\ u[v/x]$	
(renaming)	$[\alpha]\mu\beta.u$	$\triangleright\ u[\alpha/\beta]$	
(η)	$\mu\alpha.[\alpha]u$	$\triangleright\ u$	$\alpha \notin u$
(structural)	$(\mu\beta.u)v$	$\triangleright\ u\ \mu\beta.u[[\beta](wv)/[\beta]w]$	

The typed case:

hypothetical types: $\mathcal{H}p_1,\ \mathcal{H}p_2, \ldots$; (a countable sequence)
expectation types: $E \ := \ \mathcal{E}p \mid E_1 \supset E_2$

$$x : \mathcal{E}p \ ; \ \vdash x : \mathcal{E}p \ ; \ \overline{\alpha} : \Delta \qquad\qquad ; \ \alpha : \mathcal{H}p \ ; \ \alpha : \mathcal{H}p \ ; \ \overline{\alpha} : \Delta$$

$$\lambda\,\frac{\overline{x} : \Gamma, x : E_1; \vdash t : E_2; \overline{\alpha} : \Delta}{\overline{x} : \Gamma; \vdash \lambda x.t : E_1 \supset E_2; \overline{\alpha} : \Delta}\supset\text{-I}$$

$$\text{app}\,\frac{\overline{x} : \Gamma; \vdash t : E_1 \supset E_2; \overline{\alpha} : \Delta \qquad \overline{x} : \Gamma; \vdash; u : E_1; \overline{\alpha} : \Delta}{\overline{x} : \Gamma; \vdash (tu) : E_2; \alpha : \mathcal{H}p_i, \overline{\alpha} : \Delta}\supset\text{-E}$$

μ-rule and $[\alpha]$-rule are as above, section (**9**)

9.2. No typing of structural reduction here.

- We can assume that all μ-terms are typed as

$$\mu\alpha.t : \mathcal{E}p \quad \text{for} \quad t : \mathcal{H}p.$$

- such terms are normal w.r.t. *structural reduction*.

Typed structural reduction in NK
Prawitz 1965, Parigot 1990
reduces the type complexity of μ-terms.

$$\dfrac{\overset{(1)}{\beta : \neg(A \supset B)} \qquad \overset{\vdots}{w : A \supset B}}{[\beta]w : \bot}$$

$$\vdots$$

$$\dfrac{\dfrac{u = [\alpha]t : \bot}{\mu\beta.u : A \supset B}\,(1) \qquad \overset{\vdots}{v : A}}{(\mu\beta.u)v : B}$$

reduces to

$$\dfrac{\overset{(1)}{\beta : \neg B} \qquad \dfrac{\overset{\vdots}{w : A \supset B} \quad \overset{\vdots}{v : A}}{(wv) : B}}{[\beta](wv) : \bot}$$

$$\vdots$$

$$\dfrac{u[\ [\beta](wv)/[\beta]w\ ] : \bot}{\mu\beta.u[\ [\beta](wv)/[\beta]w\ ] : B}\,(1)$$

Question: what about a *linear* $\lambda\mu$?

10. Natural Deduction for Co-Intuitionism.

Multiple-conclusion single-premise ND:

sequent-style $H \vdash C_1, \ldots, C_n$

with implicit substitution, exchange, weakening and *contraction* right.

Assumptions

$$H \vdash H.$$

Subtraction

$$\smallsetminus\text{-}intro\ \frac{H \vdash \Gamma, C \qquad D \vdash \Delta}{H \vdash \Gamma, C \smallsetminus D, \Delta}$$

$$\smallsetminus\text{-}elim\ \frac{H \vdash \Delta, C \smallsetminus D \qquad C \vdash D, \Upsilon}{H \vdash \Delta, \Upsilon}$$

Normalization step for subtraction:

$$\smallsetminus\text{-E}\ \dfrac{\smallsetminus\text{-I}\ \dfrac{\begin{matrix} d_1 \\ H \vdash \Gamma, C \end{matrix} \qquad \begin{matrix} d_3 \\ D \vdash \Delta \end{matrix}}{H \vdash \Gamma, \Delta, C \smallsetminus D} \qquad \begin{matrix} d_2 \\ C \vdash D, \Upsilon \end{matrix}}{H \vdash \Gamma, \Delta, \Upsilon}$$

reduces to

$$subst\ \dfrac{subst\ \dfrac{\begin{matrix} d_1 \\ H \vdash \Gamma, C \end{matrix} \qquad \begin{matrix} d_2 \\ C \vdash D, \Upsilon \end{matrix}}{H \vdash \Gamma, D, \Upsilon} \qquad \begin{matrix} d_3 \\ D \vdash \Delta \end{matrix}}{H \vdash \Gamma, \Delta, \Upsilon}$$

Disjunction

$$\curlyvee\text{-}\mathit{intro}\ \frac{H \vdash \Gamma, C, D}{H \vdash \Gamma, C \curlyvee D}$$

$$\curlyvee\text{-}\mathit{elim}\ \frac{H \vdash \Upsilon, C \curlyvee D \qquad C \vdash \Gamma \qquad D \vdash \Delta}{H \vdash \Upsilon, \Gamma, \Delta}$$

Normalization step for disjunction:

$$\diagdown\text{-I}\ \frac{\dfrac{\begin{matrix} d_1 \\ H \vdash \Upsilon, C, D \end{matrix}}{H \vdash \Upsilon, C \curlyvee D} \qquad \begin{matrix} d_2 \\ C \vdash \Gamma \end{matrix} \qquad \begin{matrix} d_3 \\ D \vdash \Delta \end{matrix}}{H \vdash \Upsilon, \Gamma, \Delta}$$

reduces to

$$\mathit{subst}\ \frac{\mathit{subst}\ \dfrac{\begin{matrix} d_1 \\ H \vdash \Upsilon, C, D \end{matrix} \qquad \begin{matrix} d_2 \\ C \vdash \Gamma \end{matrix}}{H \vdash \Upsilon, \Gamma, D} \qquad \begin{matrix} d_3 \\ D \vdash \Delta \end{matrix}}{H \vdash \Upsilon, \Gamma, \Delta}$$

10.1. Computational interpretation.

$$\smallsetminus\textit{-intro}\,\frac{x : H \vdash \bar{t} : \Gamma, t : C \qquad y : D \vdash \overline{u} : \Delta}{x : H \vdash \bar{t} : \Gamma, \mathtt{mkc}(t, \mathtt{y}) : C \smallsetminus D, \overline{u}' : \Delta}$$

if $t : C$ and $y : D$, then $\mathtt{make{-}coroutine}(t, \mathtt{y}) : C \smallsetminus D$
but there are side effects: $\overline{u}' = u\{y := \mathtt{y}(t)\}$

$$\smallsetminus\textit{-elim}\,\frac{z : H \vdash \overline{w} : \Delta, w : C \smallsetminus D \qquad v : C \vdash s : D, \overline{s} : \Upsilon}{z : H \vdash \mathtt{postp}(v \mapsto s, w) : \bullet \mid\ \overline{w} : \Delta, \overline{s}' : \Upsilon}$$

if $w : C \smallsetminus D$, $v : C$ and $s : D$, then the term
$\mathtt{postpone}(v \mapsto s, w)$ is stored away,
but there are side effects: $\overline{s}' = \overline{s}\{v := \mathtt{v}(w)\}$.

Normalization step for subtraction:

$$\smallsetminus\text{-E}\,\frac{\smallsetminus\text{-I}\,\dfrac{\overset{d_1}{x : H \vdash \bar{t} : \Gamma, t : C} \qquad \overset{d_3}{y : D \vdash \overline{u} : \Delta}}{\begin{array}{c}x : H \vdash \bar{t} : \Gamma, \overline{u}' : \Delta,\\ \mathtt{mkc}(t, \mathtt{y}) : C \smallsetminus D\end{array}} \qquad \overset{d_2}{v : C \vdash s : D, \overline{s} : \Upsilon}}{\begin{array}{c}x : H \vdash \mathtt{postp}(v \mapsto s, \mathtt{mkc}(t, \mathtt{y})) : \bullet \mid \\ \mid \bar{t} : \Gamma, \overline{u}' : \Delta, \overline{s}' : \Upsilon\end{array}}$$

reduces to

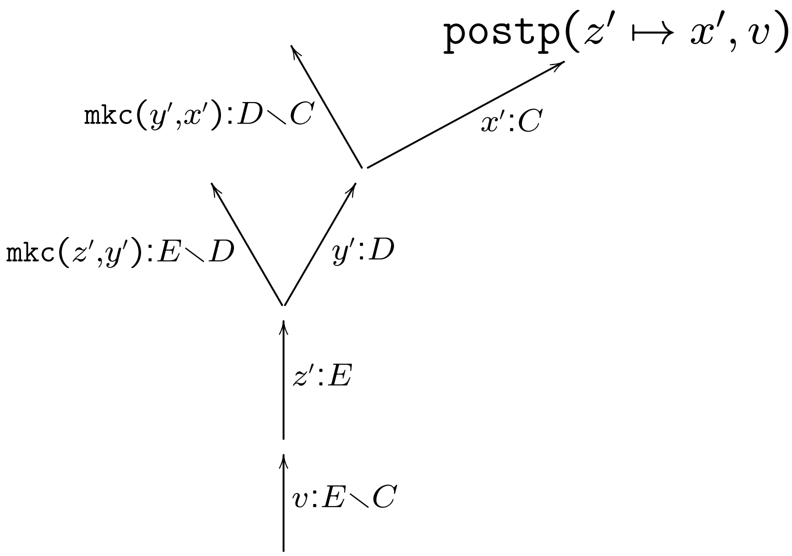
$$\textit{sub}\,\frac{\textit{sub}\,\dfrac{\overset{d_1}{x : H \vdash \bar{t} : \Gamma, t : C} \qquad \overset{d_2}{v : C \vdash s : D, \overline{s} : \Upsilon}}{\begin{array}{c}x : H \vdash \bar{t} : \Gamma, s'' : D, \overline{s}'' : \Upsilon\\ \text{[with } s'' = s\{v := t\}, \overline{s}'' = \overline{s}\{v := t\}\text{]}\end{array}} \qquad \overset{d_3}{y : D \vdash \overline{u} : \Delta}}{\begin{array}{c}x : H \vdash \bar{t} : \Gamma, \overline{u}'' : \Delta, \overline{s}'' : \Upsilon\\ \text{[with } \overline{u}'' = \overline{u}\{y := s''\}\text{]}\end{array}}$$

Example 1.

The dual of $f : C \to D, g : D \to E \vdash \lambda x.gfx : C \to E$:

$$\dfrac{v : E \setminus C \vdash v : E \setminus C \qquad \dfrac{z : E \vdash z : E \qquad \dfrac{y : D \vdash y : D \qquad x : C \vdash x : C}{y : D \vdash \mathtt{mkc}(y, \mathtt{x}) : D \setminus C, \mathtt{x}_y : C}}{z : E \vdash \mathtt{mkc}(z, \mathtt{y}) : E \setminus D, \; \mathtt{mkc}(\mathtt{y}_z, \mathtt{x}) : D \setminus C, \mathtt{x}_{\mathtt{y}z} : C}}{v : E \setminus C \vdash \mathtt{postp}(z \mapsto \mathtt{x}_{\mathtt{y}z}, v) \mid \; \mid \mathtt{mkc}(\mathtt{z}_v, \mathtt{y}) : E \setminus D, \mathtt{mkc}(\mathtt{y}_{\mathtt{z}v}, \mathtt{x}) : D \setminus C}$$

A graphical notation:



$$x' = \mathtt{x}_{\mathtt{yz}v}$$

$$y' = \mathtt{y}_{\mathtt{z}v}$$

$$z' = \mathtt{z}_v$$

- Here $\mathtt{x}_{\mathtt{yz}v} = \mathtt{x}(\mathtt{y}(\mathtt{z}(v)))$, $\mathtt{y}_{\mathtt{z}v} = \mathtt{y}(\mathtt{z}(v)))$, $\mathtt{z}_v = \mathtt{z}(v)$ are *"Herbrand terms"* expressing "remote binding", that is induced by terms of the forms `make−coroutine` and `postpone`.

- *A concurrent calculus*, "distributed" over multiple conclusions. It has been translated into λP membrane computing [Bellin & Menti 2014].

10.2. Co-intuitionistic Term assignment.
(*Linear case*)

Fvars: a countable set of *free variables* x, y, z, ...;
Funct: a countable set of *unary functions* $\mathtt{x}$, $\mathtt{y}$, $\mathtt{z}$, ...
Terms:

$$t, u \; := x \mid \mathtt{x}(t) \mid t \wp u \mid \mathtt{casel}(t) \mid \mathtt{caser}(t) \mid \mathtt{mkc}(t, \mathtt{x})$$

Trm: an enumeration of the terms t_1, t_2, ... freely generated from a variable a, with a fixed bijection $f : \mathbf{Trm} \to \mathbf{Vars}\ t_i \mapsto x_i$ [needed to restore free variables for the bound ones].
Pterms: $\mathtt{postp}(\mathtt{y} \mapsto u\{y := \mathtt{y}(t)\}, t)$, with t is a term and u is a term [such that y occurs in u (*linearity*)].
Computational context $\mathcal{S}_x$: set of terms containing exactly one free variable x.
Reductions: transformations $\mathcal{S}_x \rightsquigarrow \mathcal{S}'_x$ of the computational context.
Reductions: Let $\mathcal{S}_x$ have one of the forms 1-3:
1. $\mathcal{S}_x[\mathtt{casel}(t \wp u)]$ *locally reduces to* $\mathcal{S}_x[t]$.
2. $\mathcal{S}_x[\mathtt{caser}(t \wp u)]$ *locally reduces to* $\mathcal{S}_x[u]$.
3. $\mathcal{S}_x[\mathtt{postp}(z \mapsto u, \mathtt{mkc}(t, \mathtt{y}))]$: given a partition

$$\mathcal{S}_x[\;] \;=\; \kappa,\; \overline{\zeta}_{\mathtt{y}}\; \overline{\xi}_{\mathtt{z}}$$

where
- $\overline{\xi}_{\mathtt{z}} = \overline{\xi}_z\{z := \mathtt{z}(\mathtt{mkc}(t, \mathtt{y}))\}$;
- $\overline{\zeta}_{\mathtt{y}} = \overline{\zeta}_y\{y := \mathtt{y}(t)\}$;
- $\overline{\kappa}$ contains neither $\mathtt{z}$ nor $\mathtt{y}$,

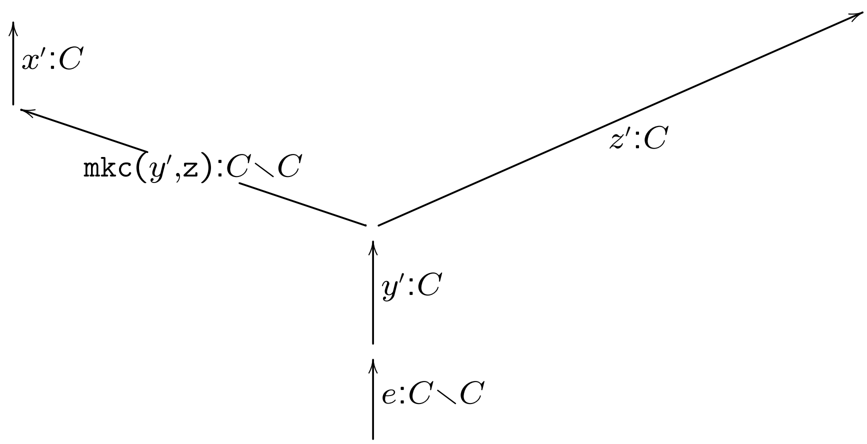
$\mathcal{S}_x$ *globally reduces* to

$$\overline{\kappa}, \quad \overline{\zeta}_y\{y := u\{z := t\}\}, \quad \overline{\xi}_z\{z := t\}.$$

Example 2.

The dual of $\vdash \lambda y.(\lambda x.x)y : C \to C \;\rightsquigarrow \vdash \lambda y.y : C \to C$:

$\mathcal{S}'$: $\quad \mathtt{postp}(x' \mapsto x', \mathtt{mkc}(y, \mathtt{z}) \quad \mathtt{postp}(y' \mapsto z', e)$



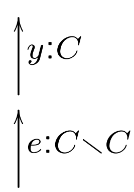
$x' = \mathtt{x}_{\mathtt{mkc}(y',\mathtt{z})}$

$z' = \mathtt{z}_{\mathtt{y}e}$

$y' = \mathtt{y}_e$

reduces to

$\mathcal{S}'$: $\quad \mathtt{postp}(y \mapsto y, e)$



Non linear case: Use lists of terms to handle weakening and contraction right. We need

$$\ell \;=: [] \mid [t] \mid \ell * \ell \qquad \text{where } * \text{ is } \mathtt{append}.$$

in terms $\mathtt{postpone}(x \mapsto \ell, t)$.

10.3. Work in progress. A probabilistic model?
To formulas H, $C_1, \ldots, C_n$ we assign *events* $\mathbf{H}, \mathbf{C_1}, \ldots, \mathbf{C_n}$ $\mathbf{H} \neq \emptyset$ *in a probability space*. We would like to read

$$H \vdash C_1, \ldots, C_n$$

as $Pr(\mathbf{C_1} \cup \ldots \cup \mathbf{C_n} | \mathbf{H}) = 1$.
Decomposition Lemma. *Let d be a Natural Deduction derivation of $H \vdash C_1, \ldots, C_n$. There are pairwise independent events $\mathbf{C'_1} \subseteq \mathbf{C_1}, \ldots, \mathbf{C'_n} \subseteq \mathbf{C_n}$ such that*

$$(\mathbf{C'_1} \cup \ldots \cup \mathbf{C'_n}) \cap \mathbf{H} = \mathbf{H}.$$

This allows us to consider also $H = 0$.
Proof. By induction on d.
- assumption $H \vdash H$: obvious.
- substitution: immediate from the ind. hyp.

$$\textit{subtraction-intro}\,\frac{H \vdash \Gamma, C \qquad D \vdash \Delta}{H \vdash \Gamma, C \smallsetminus D, \Delta}$$

- suppose $((\bigcup \mathbf{\Gamma}) \cup \mathbf{C}) \cap \mathbf{H} = \mathbf{H}$ and $(\bigcup \mathbf{\Delta}) \cap \mathbf{D} = \mathbf{D} \neq \emptyset$, where events in $\mathbf{\Gamma}$ are pairwise independent. Then $\mathbf{C} = (\mathbf{C} \cap \overline{\mathbf{D}}) \cup (\mathbf{C} \cap \mathbf{D}) = (\mathbf{C} \cap \overline{\mathbf{D}}) \cup (\mathbf{C} \cap \mathbf{D} \cap (\bigcup \mathbf{\Delta}))$, hence $\mathbf{C} \cap \mathbf{H} = [(\mathbf{C} \cap \overline{\mathbf{D}}) \cap \mathbf{H}] \cup [\mathbf{C} \cap \mathbf{D} \cap (\bigcup \mathbf{\Delta}) \cap \mathbf{H}]$.
Let $\mathbf{D'} = (\mathbf{D_j} \cap \mathbf{C} \cap \mathbf{D}) \subseteq \mathbf{D_j} \in \mathbf{\Delta}$. Then

$$\mathbf{H} = ((\cup_\mathbf{i} \mathbf{C_i}) \cup (\mathbf{C} \cup \overline{D} \cup (\cup_\mathbf{j} \mathbf{D'_j})) \cap \mathbf{H} = \mathbf{H}.$$

- subtraction elim: *supposing* $\mathbf{D}$ *and* $\mathbf{Y_i} \in \mathbf{\Upsilon}$ *pairwise independent*, and $(\mathbf{D} \cup (\bigcup \mathbf{\Upsilon})) \cap \mathbf{C} = \mathbf{C}$, then $(\bigcup \mathbf{\Upsilon}) \cap \mathbf{C} \cap \overline{\mathbf{D}} = \mathbf{C} \cap \overline{\mathbf{D}}$.

$$\curlyvee\text{-}elim \frac{H \vdash \Upsilon, C \curlyvee D \quad C \vdash \Gamma \quad D \vdash \Delta}{H \vdash \Upsilon, \Gamma, \Delta}$$

- disjunction elim: Suppose $(\mathbf{\Upsilon} \cap \mathbf{H}) \cup ((\mathbf{C} \cup \mathbf{D}) \cap \mathbf{H}) = \mathbf{H}$.
We cannot suppose $\mathbf{C}$ and $\mathbf{D}$ to be independent events; if $\mathbf{C} \cap \mathbf{D} \neq \emptyset$, then let $\mathbf{\Gamma} = \mathbf{\Gamma_0} \cup \mathbf{\Gamma_1}$ where
- $\mathbf{\Gamma_0} = \{\mathbf{C_i} \cap \overline{\mathbf{D}} : \mathbf{C_i} \in \mathbf{\Gamma}\}$ and $\mathbf{\Gamma_1} = \{\mathbf{C_i} \cap \mathbf{D} : \mathbf{C_i} \in \mathbf{\Gamma}\}$;
- $\mathbf{C_0} = \mathbf{C} \cap \overline{\mathbf{D}}$.

Then $\mathbf{C_0} = (\cup \mathbf{\Gamma_0}) \cap \mathbf{C_0}$ and $\mathbf{D} = (\cup \mathbf{\Delta}) \cap \mathbf{D}$. Hence

$$\mathbf{C} \cup \mathbf{D} = \mathbf{C_0} \cup \mathbf{D} = [(\cup \mathbf{\Gamma_0}) \cap \mathbf{C_0}] \cup [(\cup \mathbf{\Delta}) \cap \mathbf{D}].$$

Set $\mathbf{\Gamma}' = \{\mathbf{C_i} \cap \mathbf{C_0} : \mathbf{C_i} \in \mathbf{\Gamma_0}\}$ and $\mathbf{\Delta}' = \{\mathbf{D_j} \cap \mathbf{D} : \mathbf{D_j} \in \mathbf{\Delta}\}$.
Notice that $\mathbf{C_i} \in \mathbf{\Gamma}'$ and $\mathbf{D_j} \in \mathbf{\Delta}'$ are pairwise disjoint.
Hence

$$(\mathbf{\Upsilon} \cap \mathbf{H}) \cup ([\ (\cup \mathbf{\Gamma}') \cup (\cup \mathbf{\Delta}')\ ] \cap \mathbf{H}) = \mathbf{H}$$

[We could have split $\mathbf{D} \vdash \mathbf{\Delta}$ instead of $\mathbf{C} \vdash \mathbf{\Gamma}$].

The case of disjunction right is immediate from the inductive hypothesis. **Qed.**

Possible connection: Lukasiewicz' many valued logic, MV-algebras, (Chang, D.Mundici).
Our Decomposition Lemma may correspond to Riesz Decomposition Theorem for Effect Algebras. [Bennett and Foulis 1995]

Let us assign probabilities to *decorated sequents* $x : C \vdash \overline{u} : \Delta$.

We claim that for any $D_j \in \Delta$ the explicit dependencies in the term $\mathtt{u}_{t_1 \ldots t_n x} : D_j$ indicate how to assign a probability to D so that all conclusions have independent assignments.
Indeed $u_{\bar{t}tx} : D_j$ arises from $u_{\bar{t}y} : D_j$ by a substitution $u_{\bar{t}y}\{y := t_x\}$ where

- either $t_x : C$ and $y : D$ are premises of a $\setminus$-intro with conclusion $\mathtt{mkc}(t, \mathtt{y}) : C \setminus D$,
- or $t_x : C \setminus D$ is a major premise of a $\setminus$-elim, and $y : C$ is the only free variable in the computational environment of the minor premise, which D_j belongs to.

In both cases the new term t_x signals that we need to decompose the event $\mathbf{D_j}$ by taking the intersection $\mathbf{C} \cap \mathbf{D_j}$ or $(\mathbf{C} \cap \overline{\mathbf{D}}) \cap \mathbf{D_j}$, as in the proof above.

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