Directed containers as categories

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Motivation

- Containers are a representation of a wide class of set functors (datatypes) in terms of shapes and positions.
- Containers are a great tool for doing combinatorics of datatypes.
- Polynomials are essentially the same as containers.
- Directed containers (A., Chapman, U., FoSSaCS 2012) are a specialization of containers to those whose interpretation is a comonad.
  In a directed container, a positions of shape defines another shape (the subshape).

This talk

- We look at directed containers through the polynomial glasses.
- This reveals a symmetry in directed containers (between shapes and subshapes) that is absent in directed container morphisms.
- A directed container is a category.
  But a directed container morphism is not a functor.
- We consider two basic constructions/specializations that this identification suggests.

Plan

- Containers and directed containers
- The polynomial view
- The opposite directed container
- Bidirected containers

Containers

- A *container* is a set S (of shapes) and a S-indexed family of sets P (of positions).
- It interprets into a set functor $[\![S, P]\!]^{\mathrm{c}} =_{\mathrm{df}} F$ where

$$F\,X =_{\mathrm{df}} \Sigma s : S.\, P\,s \to X$$
$$F\,f\,(s, v) =_{\mathrm{df}} (s, \lambda p.\, f\,(v\,p))$$

- A *container morphism* between (S, P) and (S', P') is given by maps $t : S \to S'$ and $q : \Pi s : S.\, P'\,(t\,s) \to P\,s$.
- It interprets into a nat. tr. $[\![t, q]\!]^{\mathrm{c}} =_{\mathrm{df}} \tau : [\![S, P]\!]^{\mathrm{c}} \to [\![S', P']\!]^{\mathrm{c}}$ where

$$\tau\,(s, v) =_{\mathrm{df}} (t\,s, \lambda p.\, v\,(q\,s\,p))$$

- Containers and container morphisms form a monoidal category **Cont**.
- Interpretation $[\![-]\!]^{\mathrm{c}}$ is a fully faithful monoidal functor from **Cont** to $[\mathbf{Set}, \mathbf{Set}]$.

Directed containers

- A *directed container* is a container (S, P) with
  - $\downarrow : \Pi s : S.\, P\, s \to S$ (the subshape for a position),
  - $\mathrm{o} : \Pi\{s : S\}.\, P\, s$ (the root position),
  - $\oplus : \Pi\{s : S\}.\, \Pi p : P\, s.\, P\,(s \downarrow p) \to P\, s$ (translation of subshape positions)

  such that

$$
\begin{aligned}
s \downarrow \mathrm{o} &= s \\
s \downarrow (p \oplus p') &= (s \downarrow p) \downarrow p' \\
p \oplus \mathrm{o} &= p \\
\mathrm{o} \oplus p &= p \\
(p \oplus p') \oplus p'' &= p \oplus (p' \oplus p'')
\end{aligned}
$$

- It interprets into a set comonad $[\![S, P, \downarrow, \mathrm{o}, \oplus]\!]^{\mathrm{dc}} =_{\mathrm{df}} (D, \varepsilon, \delta)$ where

$$
\begin{aligned}
D &=_{\mathrm{df}} [\![S, P]\!]^{\mathrm{c}} \\
\varepsilon\,(s, v) &=_{\mathrm{df}} v\,(\mathrm{o}\,\{s\}) \\
\delta\,(s, v) &=_{\mathrm{df}} (s, \lambda p.\,(s \downarrow p, \lambda p'.\, v\,(p \oplus \{s\}\, p')))
\end{aligned}
$$

Directed containers ctd

- A *directed container morphism* between $(S, P, \downarrow, \mathsf{o}, \oplus)$ and $(S', P', \downarrow', \mathsf{o}', \oplus')$ is a container morphism (t, q) between (S, P) and (S', P') such that

$$
\begin{aligned}
t\,(s \downarrow q\,s\,p) &= t\,s \downarrow' p \\
\mathsf{o}\,\{s\} &= q\,s\,(\mathsf{o}'\,\{t\,s\}) \\
q\,s\,p \oplus \{s\}\,q\,(s \downarrow q\,s\,p)\,p' &= q\,s\,(p \oplus' \{t\,s\}\,p')
\end{aligned}
$$

- It interprets into $[\![t, q]\!]^{\mathrm{dc}} =_{\mathrm{df}} [\![t, q]\!]^{\mathrm{c}}$, which is a comonad morphism betw. $[\![S, P, \downarrow, \mathsf{o}, \oplus]\!]^{\mathrm{dc}}$ and $[\![S', P', \downarrow', \mathsf{o}', \oplus']\!]^{\mathrm{dc}}$.
- Directed containers and directed container morphisms form a category **DCont**.
- Interpretation $[\![-]\!]^{\mathrm{dc}}$ is a fully-faithful functor from **DCont** to **Comonads**(**Set**).
- In fact $[\![-]\!]^{\mathrm{dc}}$ is the pullback of $[\![-]\!]^{\mathrm{c}}$ along $U : \mathbf{Comonads}(\mathbf{Set}) \to [\mathbf{Set}, \mathbf{Set}]$.

Streams, lists with suffixes, lists with cyclic shifts

- Streams:

  $S =_{\mathrm{df}} 1,\ P * =_{\mathrm{df}} \mathsf{Nat},\ * \downarrow p =_{\mathrm{df}} *,$
  $\mathsf{o} =_{\mathrm{df}} 0,\ p \oplus p' =_{\mathrm{df}} p + p'$

  $D\,X =_{\mathrm{df}} \Sigma * : 1.\, \mathsf{Nat} \to X \cong \mathsf{Str}\, X$

- Lists with suffixes:

  $S =_{\mathrm{df}} \mathsf{Nat},\ P\,s =_{\mathrm{df}} [0..s],\ s \downarrow p =_{\mathrm{df}} s - p,$
  $\mathsf{o} =_{\mathrm{df}} 0,\ p \oplus p' =_{\mathrm{df}} p + p'$

  $D\,X =_{\mathrm{df}} \Sigma s : \mathsf{Nat}.\, [0..s] \to X \cong \mathsf{NEList}\, X$

- Lists with cyclic shifts:

  $S =_{\mathrm{df}} \mathsf{Nat},\ P\,s =_{\mathrm{df}} [0..s],\ s \downarrow p =_{\mathrm{df}} s,$
  $\mathsf{o} =_{\mathrm{df}} 0,\ p \oplus \{s\}\, p' =_{\mathrm{df}} (p + p') \bmod (s + 1)$

  $D\,X =_{\mathrm{df}} \Sigma s : \mathsf{Nat}.\, [0..s] \to X \cong \mathsf{NEList}\, X$

Reader comonad, array comonad

- Reader comonad:

  S any given set, $P\,s =_{\mathrm{df}} 1$

  $D\,X =_{\mathrm{df}} \Sigma s : S.\, 1 \to X \cong S \times X$

- Array (costate) comonad:

  S any given set, $P\,s =_{\mathrm{df}} S$, $s \downarrow s' =_{\mathrm{df}} s'$,
  $\mathsf{o}\,\{s\} =_{\mathrm{df}} s$ and $s' \oplus \{s\}\, s'' =_{\mathrm{df}} s''$

  $D\,X =_{\mathrm{df}} \Sigma s : S.\, S \to X \cong S \times (S \to X)$

Containers as polynomials

- A *polynomial* is given by sets S and $\bar{P}$ (positions across all shapes) and a map $\mathbf{s} : \bar{P} \to S$ (the shape of a position).
- Polynomials are in a bijection up to iso. with containers. They are interconverted by

$$\bar{P} =_{\mathrm{df}} \Sigma s : S.\, P\, s \qquad\qquad P\, s =_{\mathrm{df}} \Sigma p : \bar{P}.\, \{\mathbf{s}\, p = s\}$$

$$\mathbf{s}\,(s, p) =_{\mathrm{df}} s$$

Containers as polynomials ctd

- A *polynomial morphism* between $(S, \bar{P}, \mathbf{s})$ and $(S', \bar{P}', \mathbf{s}')$ is given by maps $t : S \to S'$ and $\bar{q} : (\Sigma s : S. \Sigma p : \bar{P}'. \{t\, s = \mathbf{s}'\, p\}) \to \bar{P}$ such that $\mathbf{s}\,(\bar{q}\,(s, p)) = s$.
- Container morphisms and polynomial morphisms are interconverted by

$$\bar{q}\,(s, p) =_{\mathrm{df}} q\, s\, p \qquad\qquad q\, s\, p =_{\mathrm{df}} \bar{q}\,(s, p)$$

- Polynomials and polynomial morphisms form category **Poly**.
- **Cont** and **Poly** are equivalent categories.

Directed containers as "directed polynomials"

- A *directed polynomial* is given by sets S, $\bar{P}$, a map $\mathbf{s} : \bar{P} \to S$ and maps
    - $\mathbf{t} : \bar{P} \to S$,
    - $\mathbf{id} : \{S\} \to \bar{P}$ such that $\mathbf{s}\,(\mathbf{id}\,\{s\}) = s$,
    - $; : (\Sigma p : \bar{P}.\, \Sigma p' : \bar{P}.\, \{\mathbf{t}\, p = \mathbf{s}\, p'\}) \to \bar{P}$ such that $\mathbf{s}\,(p \,;\, p') = \mathbf{s}\, p$

  such that

$$\begin{aligned} \mathbf{t}\,(\mathbf{id}\,\{s\}) &= s \\ \mathbf{t}\,(p \,;\, p') &= \mathbf{t}\, p' \\ p \,;\, \mathbf{id}\,\{s\} &= p \\ \mathbf{id}\,\{s\} \,;\, p &= p \\ (p \,;\, p') \,;\, p'' &= p \,;\, (p' \,;\, p'') \end{aligned}$$

  i.e., a category!

- Directed containers and directed polynomials in a bijection up to iso.; they are interconverted by

$$\begin{array}{rcl} \bar{P} =_{\mathrm{df}} \Sigma s : S.\, P\, s & \qquad & P\, s =_{\mathrm{df}} \Sigma p : \bar{P}.\, \{\mathbf{s}\, p = s\} \\ \mathbf{s}\,(s, p) =_{\mathrm{df}} s & & \\ \mathbf{t}\,(s, p) =_{\mathrm{df}} s \downarrow p & & s \downarrow p =_{\mathrm{df}} \mathbf{t}\, p \\ \mathbf{id}\,\{s\} =_{\mathrm{df}} (s, \mathsf{o}\,\{s\}) & & \mathsf{o}\,\{s\} =_{\mathrm{df}} \mathbf{id}\,\{s\} \\ (s, p)\,;(s \downarrow p, p') =_{\mathrm{df}} (s, p \oplus \{s\}\, p') & & p \oplus \{s\} p' =_{\mathrm{df}} p\,; p' \end{array}$$

Directed containers as "directed polynomials" ctd

- A *directed polynomial morphism* between $(S, \bar{P}, \mathbf{s}, \mathbf{t}, \mathbf{id}, ;)$ and $(S', \bar{P}', \mathbf{s}', \mathbf{t}', \mathbf{id}', ;')$ is given by maps $t : S \to S'$ and $\bar{q} : (\Sigma s : S. \Sigma p : \bar{P}'. \{t\, s = \mathbf{s}'\, p)\} \to \bar{P}$ such that $\mathbf{s}\,(\bar{q}\,(s, p)) = s$ and

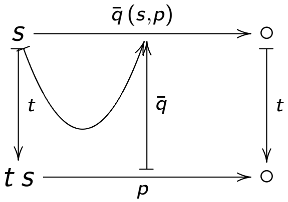
$$t\,(\mathbf{t}\,(\bar{q}\,(s, p))) = \mathbf{t}'\, p$$
$$\mathbf{id}\,\{s\} = \bar{q}\,(s, \mathbf{id}'\,\{t\, s\})$$
$$\bar{q}\,(s, p)\,;\,\bar{q}\,(\mathbf{t}\,(\bar{q}(s, p)), p') = \bar{q}\,(s, p\,;'\, p')$$

- This is nothing like a functor. At best we could call it a "relative split pre-opcleavage".
  t is only an object map, not a functor!
- Directed polynomials forma catecory **DPoly**.
- **DCont** and **DPoly** are equivalent categories. Directed container morphisms and directed polynomial morphisms are interconverted by

$$\bar{q}\,(s, p) =_{\mathrm{df}} q\, s\, p \qquad\qquad q\, s\, p =_{\mathrm{df}} \bar{q}\,(s, p)$$

Directed containers as “directed polynomials” ctd

- For $(t, \bar{q})$ a directed polynomial morphism between $E = (S, \bar{P}, \mathbf{s}, \mathbf{t}, \mathbf{id}, ;)$ and $E' = (S', \bar{P}', \mathbf{s}', \mathbf{t}', \mathbf{id}', ;')$ we have



- We could reasonably say that $\bar{q}$ is a split pre-opcleavage for $t^\dagger : E \to S'^\dagger$ relative to $! : E' \to S'^\dagger$ where $S'^\dagger$ is the cofree category on S'.
- “pre” — we don’t require the maps delivered to be opCartesian;
  “split” — we do require them to agree with each other

Streams, lists w. suffixes, lists w. cyclic shifts again

- Streams:

  $S =_{\mathrm{df}} 1,\ \bar{P} =_{\mathrm{df}} \mathsf{Nat},\ \mathbf{s}\, p =_{\mathrm{df}} *,$
  $\mathbf{t}\, p =_{\mathrm{df}} *,\ \mathbf{id} =_{\mathrm{df}} 0,\ p \,;\, p' =_{\mathrm{df}} p + p'$

  This is the monoid $(\mathsf{Nat}, 0, +)$ seen as a category.

- Lists with suffixes:

  $S =_{\mathrm{df}} \mathsf{Nat},\ \bar{P} =_{\mathrm{df}} \Sigma s : \mathsf{Nat}.\, [0..s],\ \mathbf{s}\,(s, p) =_{\mathrm{df}} s,$
  $\mathbf{t}\,(s, p) =_{\mathrm{df}} s - p,\ \mathbf{id}\,\{s\} =_{\mathrm{df}} (s, 0),$
  $(s, p) \,;\, (s - p, p') =_{\mathrm{df}} (s, p + p')$

- Lists with cyclic shifts:

  $S =_{\mathrm{df}} \mathsf{Nat},\ \bar{P} =_{\mathrm{df}} \Sigma s : \mathsf{Nat}.\, [0..s],\ \mathbf{s}\,(s, p) =_{\mathrm{df}} s,$
  $\mathbf{t}\,(s, p) =_{\mathrm{df}} s,\ \mathbf{id}\,\{s\} =_{\mathrm{df}} (s, 0),$
  $(s, p) \,;\, (s, p') =_{\mathrm{df}} (s, (p + p') \bmod (s + 1))$

Reader comonad, array comonad

- Reader comonad:

  S any given set, $\bar{P} =_{\mathrm{df}} \Sigma s : S.\, 1 \cong S$, $\mathbf{s}\, s =_{\mathrm{df}} s$,
  $\mathbf{t}\, s =_{\mathrm{df}} s$, $\mathbf{id}\, \{s\} =_{\mathrm{df}} s$, $s \,;\, s =_{\mathrm{df}} s$

  This is the discrete category (free category) on S.

  $D\, X =_{\mathrm{df}} \Sigma s : S.\, 1 \to X \cong S \times X$

- Array (costate) comonad:

  S any given set, $\bar{P} =_{\mathrm{df}} \Sigma s : S.\, S \cong S \times S$, $\mathbf{s}\,(s, s') =_{\mathrm{df}} s$,
  $\mathbf{t}\,(s, s') =_{\mathrm{df}} s'$, $\mathbf{id}\, \{s\} =_{\mathrm{df}} (s, s)$, $(s, s') \,;\, (s', s'') =_{\mathrm{df}} (s, s'')$

  This is the codiscrete category (cofree category) on S.

  $D\, X =_{\mathrm{df}} \Sigma s : S.\, S \to X \cong S \times (S \to X)$

The opposite directed container

- Given a category $(S, \bar{P}, \mathbf{s}, \mathbf{t}, \mathbf{id}, ;)$, the *opposite category* is $(S^{\mathrm{op}}, \bar{P}^{\mathrm{op}}, \mathbf{s}^{\mathrm{op}}, \mathbf{t}^{\mathrm{op}}, \mathbf{id}^{\mathrm{op}}, ;^{\mathrm{op}})$ where

$$
\begin{aligned}
S^{\mathrm{op}} &=_{\mathrm{df}} S \\
\bar{P}^{\mathrm{op}} &=_{\mathrm{df}} \bar{P} \\
\mathbf{s}^{\mathrm{op}}\, p &=_{\mathrm{df}} \mathbf{t}\, p \\
\mathbf{t}^{\mathrm{op}}\, p &=_{\mathrm{df}} \mathbf{s}\, p \\
\mathbf{id}^{\mathrm{op}}\, \{s\} &=_{\mathrm{df}} \mathbf{id}\, \{s\} \\
f \;;^{\mathrm{op}}\; g &=_{\mathrm{df}} g \;;\; f
\end{aligned}
$$

- Given a directed container $(S, P, \downarrow, \mathrm{o}, \oplus)$, the *"opposite" directed container* is $(S^{\mathrm{op}}, P^{\mathrm{op}}, \downarrow^{\mathrm{op}}, \mathrm{o}^{\mathrm{op}}, \oplus^{\mathrm{op}})$ where

$$
\begin{aligned}
S^{\mathrm{op}} &=_{\mathrm{df}} S \\
P^{\mathrm{op}}\, s &=_{\mathrm{df}} \Sigma s' : S.\, \Sigma p : P\, s'.\, \{s = s' \downarrow p\} \\
s \downarrow^{\mathrm{op}} (s', p) &=_{\mathrm{df}} s' \\
\mathrm{o}^{\mathrm{op}}\, \{s\} &=_{\mathrm{df}} (s, \mathrm{o}\, \{s\}) \\
(s', p) \oplus^{\mathrm{op}} \{s\}\, (s'', p') &=_{\mathrm{df}} (s'', p' \oplus \{s''\}\, p)
\end{aligned}
$$

Lists with suffixes

- The opposite category is:

$$\begin{gathered} S^{\mathrm{op}} =_{\mathrm{df}} \mathsf{Nat} \\ \bar{P}^{\mathrm{op}} =_{\mathrm{df}} \Sigma s : \mathsf{Nat}.\, [0..s] \\ \mathbf{s}^{\mathrm{op}}\,(s, p) =_{\mathrm{df}} s - p \\ \mathbf{t}^{\mathrm{op}}\,(s, p) =_{\mathrm{df}} s \\ \mathbf{id}^{\mathrm{op}}\,\{s\} =_{\mathrm{df}} (s, 0) \\ (s - p, p')\,;^{\mathrm{op}}\,(s, p) =_{\mathrm{df}} (s, p + p') \end{gathered}$$

Lists with suffixes ctd

- The opposite directed container is (the systematic definition and a simplified one):

$$
\begin{array}{rcl}
S^{\mathrm{op}} &=_{\mathrm{df}}& \mathsf{Nat} \\
P^{\mathrm{op}}\, s &=_{\mathrm{df}}& \Sigma s' : \mathsf{Nat}.\, \Sigma p : [0..s'].\, \{s = s' - p\} \\
s \downarrow^{\mathrm{op}} (s', p) &=_{\mathrm{df}}& s' \\
\mathrm{o}^{\mathrm{op}}\, \{s\} &=_{\mathrm{df}}& (s, 0) \\
(s', p) \oplus^{\mathrm{op}} \{s\}\, (s'', p') &=_{\mathrm{df}}& (s'', p' + p)
\end{array}
\qquad
\begin{array}{rcl}
S^{\mathrm{op}} &=_{\mathrm{df}}& \mathsf{Nat} \\
P^{\mathrm{op}}\, s &=_{\mathrm{df}}& \mathsf{Nat} \\
s \downarrow^{\mathrm{op}} p &=_{\mathrm{df}}& s + p \\
\mathrm{o}^{\mathrm{op}} &=_{\mathrm{df}}& 0 \\
p \oplus^{\mathrm{op}} p' &=_{\mathrm{df}}& p' + p
\end{array}
$$

- The corresponding comonad is:
  $D^{\mathrm{op}}\, X =_{\mathrm{df}} \Sigma s : \mathsf{Nat}.\, \mathsf{Nat} \to X \cong \mathsf{Nat} \times \mathsf{Str}\, X$,

  $$
  \begin{array}{l}
  \varepsilon\, (s, xs) =_{\mathrm{df}} \mathsf{hd}\, xs, \\
  \delta\, (s, xs) =_{\mathrm{df}} (s, \delta_0\, (s, xs)) \\
  \qquad \text{where } \delta_0\, (s, xs) =_{\mathrm{df}} (s, xs) :: \delta_0(s + 1, \mathsf{tl}\, xs).
  \end{array}
  $$

Bidirected containers as groupoids

- A *groupoid* is a category $(S, \bar{P}, \mathbf{s}, \mathbf{t}, \mathbf{id}, ;)$ with a map $(-)^{-1} : \bar{P} \to \bar{P}$ such that $\mathbf{s}\,(p^{-1}) = \mathbf{t}\,p$ and

$$\begin{aligned} \mathbf{t}\,(p^{-1}) &= \mathbf{s}\,p \\ p\,;(p^{-1}) &= \mathbf{id}\,\{\mathbf{s}\,p\} \\ (p^{-1})\,;p &= \mathbf{id}\,\{\mathbf{t}\,p\} \end{aligned}$$

- A *"bidirected container"* is a directed container $(S, P, \downarrow, \mathsf{o}, \oplus)$ together with a map $\ominus : \Pi\{s : S\}.\, \Pi p : P\,s.\, P\,(s \downarrow p)$ such that

$$\begin{aligned} (s \downarrow p) \downarrow (\ominus\,\{s\}\,p) &= s \\ p \oplus \{s\}\,(\ominus\,\{s\}\,p) &= \mathsf{o}\,\{s\} \\ (\ominus\,\{s\}p) \oplus \{s \downarrow p\}\,p &= \mathsf{o}\,\{s \downarrow p\} \end{aligned}$$

Bidirected containers as groupoids ctd

- Groupoids and bidirected containers are in a bijection up to iso.; the conversions are

$$\ominus \{s\}\, p =_{\mathrm{df}} p^{-1} \qquad\qquad (s, p)^{-1} =_{\mathrm{df}} (s \downarrow p, \ominus \{s\}\, p)$$

- If a category is a groupoid, it is isomorphic to its opposite category. The converse does generally not hold.

Lists with cyclic shifts

- The category fo the lists with cyclic shifts comonad is a groupoid:

  $(s, p)^{-1} =_{\mathrm{df}} (s, -p \bmod (s+1))$

  In the corresponding bidirected container we have

  $\ominus \{s\}\, p =_{\mathrm{df}} -p \bmod (s+1)$

Takeaway

- The polynomial view reveals a symmetry between shapes/subshapes in a directed container.
- This makes specific constructions and specializations available for containers that are comonads.
- Directed container morphisms do not exhibit the same symmetry.
- Directed containers appear special in that, e.g., containers that are monads do not admit a comparably simple description.