Common Knowledge

The Coinductive Formulation

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Outline

An Epistemic Puzzle

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- But it generates and communicates information about the epistemic state of all agents
- Now everybody knows that everybody knows that someone has a green dot
- ... and everybody knows that everybody knows that everybody knows ...
- There is COMMON KNOWLEDGE of the statement *"someone has a green dot"*

Common Knowledge Coinductively

We express this by a *coinductive operator*

$$\begin{array}{l}\text{CoInductive cCK} : \text{Event} \to \text{Event} \\ \quad \text{cCK}-\text{intro} : \forall e. \text{EK}\, e \sqcap \text{cCK}\,(\text{EK}\, e) \subset \text{cCK}\, e\end{array}$$

- An *event* e is any statement about the world
- $\text{EK}\, e$ means "Everybody knows e"
- $\text{cCK}\, e$ means "e is common knowledge"
- $\sqcap$ is the conjunction (intersection) of two events
- $\subset$ is the implication (inclusion) between events

$\text{cCK}\, e$: Everybody knows e and it is common knowledge that everybody knows e (corecursively)

Possible Worlds

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- Implication between events

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- An event is true in all states

$$\forall\!\!\!\forall : \text{Event} \to \text{Set} \qquad e_1 \subset e_2 = \forall\!\!\!\forall\, (e_1 \sqsubset e_2)$$
$$\forall\!\!\!\forall\ e = \forall w. e\, w$$

Knowledge Operators

Agents

A type of Agents

Every a : Agent is equipped with a KNOWLEDGE OPERATOR

$$\mathsf{K}_a : \text{Event} \to \text{Event}$$

If e : Event, $\mathsf{K}_a\, e$ means "*a* knows that *e* is true (in the present state)"

To properly represent knowledge, K_a must satisfy some properties corresponding to the axiom of the system of *epistemic logic* S5

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- The Socratic Principle (Axiom 5):

$$\sim \mathsf{K}\, e \subset \mathsf{K}\,(\sim \mathsf{K}\, e)$$

Semantic Entailment

We discovered that the standard relational semantics of epistemic logic also satisfies and infinitary deduction rule

PRESERVATION OF SEMANTIC ENTAILMENT

Semantic entailment between a family $E : X \to \mathsf{Event}$ and an $e : \mathsf{Event}$

$$E \subset e = \forall w.(\forall x : X.E\,x\,w) \to e\,w$$

Knowledge preserves semantic entailment:

$$E \subset e \to \mathsf{K}\,E \subset \mathsf{K}\,e$$

where $\mathsf{K}\,E$ means knowledge of all events in E:

$$\mathsf{K}\,E : X \to \mathsf{Event}$$
$$\mathsf{K}\,E\,x = \mathsf{K}\,(E\,x)$$

Everybody Knows and Common Knowledge

- Everybody Knows e

$$
\begin{aligned}
&\mathrm{EK} : \mathrm{Event} \rightarrow \mathrm{Event} \\
&\mathrm{EK}\, e = \lambda w. \forall a. \mathrm{K}_a\, e\, w
\end{aligned}
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- cCK is a knowledge operator (EK isn't)
  It is as if there were an agent that knows exactly the events that are common knowledge among all agents

The Relational Semantics

Equivalences on States

- Traditional semantics of epistemic logic:
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- The two constructions are inverse of each other

Relational Common Knowledge

The traditional definition of common knowledge

$\simeq_a$ an equivalence for each agent a

Common knowledge is the transitive closure of the union of all the $\simeq_a$s

$$\begin{array}{l} \text{Inductive } \_ \propto \_ : \text{State} \to \text{State} \to \text{Set} \\ \quad \propto\!-\text{base} : \forall a. \forall w. \forall v. w \simeq_a v \to w \propto v \\ \quad \propto\!-\text{trans} : \forall w. \forall v. \forall u. w \propto v \to v \propto u \to w \propto u \end{array}$$

The coinductive and relational definitions of common knowledge are equivalent

$$\text{cCK} \equiv \text{K}_{\propto} \qquad\qquad \propto \equiv \simeq_{\text{cCK}}$$

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- Common Knowledge is a Knowledge Operator
- Equivalence with the Relational Definition

Questions

THANK YOU!

Article and Coq formalization at

http://www.duplavis.com/venanzio/