Infinite sets that satisfy the principle of omniscience in constructive type theory

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Mathematics in dependent type theory

1. I'll work in intensional Martin-Löf type theory (MLTT).

2. I will make a number of remarks related to HoTT, in particular regarding -1-truncation and equivalence.

3. Sometimes I will use *function extensionality*.

   (Alternatively, I can assume that our hypothetical functions are extensional in a suitable sense, like Bishop did. However, this leads to the so-called *setoid hell*.)

4. I will work informally but rigorously.

   But I have also written formal versions of the proofs in the computer in Agda notation.

LPO

For any given $p : \mathbb{N} \to 2$, we can either find $n : \mathbb{N}$ with $p(n) = 0$, or else determine that $p(n) = 1$ for all $n : \mathbb{N}$.

$$\Pi(p : \mathbb{N} \to 2), (\Sigma(n : \mathbb{N}), p(n) = 0) + (\Pi(n : \mathbb{N}), p(n) = 1)$$

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For any given $p : \mathbb{N} \to 2$, we can either find a root of p, or else determine that there is none.

$$\Pi(p : \mathbb{N} \to 2), (\Sigma(n : \mathbb{N}), p(n) = 0) + \neg(\Sigma(n : \mathbb{N}), p(n) = 0)$$

Subsingleton version of LPO

Any $p : \mathbb{N} \to 2$ either has a root or it doesn't.

$$\Pi(p : \mathbb{N} \to 2), \|\Sigma(n : \mathbb{N}), p(n) = 0\| + \neg(\Sigma(n : \mathbb{N}), p(n) = 0)$$

No need to singleton-truncate the rightmost Σ, as the negation of a type is automatically a subsingleton.

Also, this truncation is definable in MLTT (by considering the existence of a minimal root).

The LPO types

$$\Pi(p : \mathbb{N} \to 2), (\Sigma(n : \mathbb{N}), p(n) = 0) + \neg(\Sigma(n : \mathbb{N}), p(n) = 0)$$

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are logically equivalent, but not necessarily isomorphic
(or homotopically equivalent).

The second is a retract of the first.

(This doesn't use the HoTT formulation of the axiom of choice.)

(It is an instance of choice that just holds.)

LPO is undecided

$$\Pi(p : \mathbb{N} \to 2), (\Sigma(n : \mathbb{N}), p(n) = 0) + (\neg\Sigma(n : \mathbb{N}), p(n) = 0)$$

1. A meta-theorem is that MLTT doesn't inhabit LPO or $\neg$LPO.
2. Each of them is consistent with MLTT.

   Classical models validate LPO.

   Effective and continuous models validate $\neg$LPO.
3. LPO is undecided, and we'll keep it that way.
4. But we'll say it is a constructive taboo.

We now make $\mathbb{N}$ larger by adding a point at infinity

Let $\mathbb{N}_\infty$ be the type of decreasing binary sequences.

$$\mathbb{N}_\infty \stackrel{\text{def}}{=} \Sigma(\alpha : 2^{\mathbb{N}}), \Pi(n : \mathbb{N}), \alpha(n) = 0 \to \alpha(n+1) = 0.$$

Side-remark:

1. $\mathbb{N}$ is the *initial algebra* of the functor $1 + (-)$.

2. $\mathbb{N}_\infty$ is the *final coalgebra* of this functor.

(This requires function extensionality.)

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1. The type $\mathbb{N}$ embeds into $\mathbb{N}_\infty$ by mapping the number $n : \mathbb{N}$ to the sequence $\underline{n} \stackrel{\text{def}}{=} 1^n 0^\omega$.
2. A point not in the image of this is $\infty \stackrel{\text{def}}{=} 1^\omega$.
3. The assertion that every point of $\mathbb{N}_\infty$ is of one of these two forms is equivalent to LPO.
4. What is true is that no point of $\mathbb{N}_\infty$ is different from all points of these two forms.
5. The embedding $\mathbb{N} + 1 \to \mathbb{N}_\infty$ is an isomorphism iff LPO holds.
6. But the complement of its image is empty. We say it is dense.

Theorem

$$\Pi(p : \mathbb{N}_\infty \to 2), (\Sigma(x : \mathbb{N}_\infty), p(x) = 0) + \neg\Sigma(x : \mathbb{N}_\infty), p(n) = 0$$

1. This is LPO with $\mathbb{N}$ replaced by $\mathbb{N}_\infty$.

2. We don't use continuity axioms, which anyway are not available in MLTT.

3. However, this is motivated by topological (not homotopical) considerations.

   In Johnstone's *topological topos*, $\mathbb{N}_\infty$ gets interpreted as the one-point compactification of discrete $\mathbb{N}$.

   Here we are seeing a *logical manifestation* of *topological compactness*.

4. This theorem actually makes sense in any variety of constructive mathematics (JSL 2013).

WLPO is also undecided by MLTT

$$\Pi(p : \mathbb{N} \to 2), (\Pi(n : \mathbb{N}), p(n) = 1) + \neg\Pi(x : \mathbb{N}), p(n) = 1$$

(This implies that every Turing machine carries on for ever or it doesn't.)

But we have:

Theorem $\Pi(p : \mathbb{N}_\infty \to 2), (\Pi(n : \mathbb{N}), p(\underline{n}) = 1) + \neg\Pi(n : \mathbb{N}), p(\underline{n}) = 1$

1. The point is that now we quantify over $\mathbb{N}$, although the function p is defined on $\mathbb{N}_\infty$.
2. This again holds in any variety of constructive mathematics and doesn't rely on continuity axioms (JSL'2013).

Some consequences

1. Every function $f : \mathbb{N}_\infty \to \mathbb{N}$ is constant or not.

2. Any two functions $f, g : \mathbb{N}_\infty \to \mathbb{N}$ are equal or not.

3. Any function $f : \mathbb{N}_\infty \to \mathbb{N}$ has a minimum value, and it is possible to find the point at which the minimum value is attained.

4. For any function $f : \mathbb{N}_\infty \to \mathbb{N}$ there is a point $x : \mathbb{N}_\infty$ such that if f has a maximum value, the maximum value is x.

5. Any function $f : \mathbb{N}_\infty \to \mathbb{N}$ is not continuous, or not-not continuous.

6. There is a non-continuous function $f : \mathbb{N}_\infty \to \mathbb{N}$ iff WLPO holds.

Are there more types like $\mathbb{N}_\infty$?

1. Plenty.
2. Our business here is how to construct them.

What have we been doing?

Giving examples of types X and properties P of X such that the assertion

for all $x : X$, *either* $P(x)$ *or not* $P(x)$

just holds.

1. In classical mathematics, we assume excluded middle.
2. Here we investigate mathematically how much of it just holds.

Two notions

Definition (Omniscient type)

A type X is omniscient if for every $p : X \to 2$, the assertion that we can find $x : X$ with $p(x) = 0$ is decidable.

In symbols:

$$\Pi(p : X \to 2), (\Sigma(x : X), p(x) = 0) + (\neg\Sigma(x : X), p(x) = 0).$$

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Definition (Searchable type)
A type X is searchable if for every $p : X \to 2$ we can find $x_0 : X$, called a *universal witness* for p, such that if $p(x_0) = 1$, then $p(x) = 1$ for all $x : X$.

In symbols,

$$\Pi(p : X \to 2), \Sigma(x_0 : X), p(x_0) = 1 \to \Pi(x : X), p(x) = 1.$$

Their relationship

$$\text{omniscient}(X) \stackrel{\text{def}}{=} \Pi(p : X \to 2), (\Sigma(x : X), p(x) = 0) + (\neg\Sigma(x : X), p(x) = 0)$$

$$\text{searchable}(X) \stackrel{\text{def}}{=} \Pi(p : X \to 2), \Sigma(x_0 : X), p(x_0) = 1 \to \Pi(x : X), p(x) = 1.$$

NB. These types are not subsingletons in general.

Proposition A type X is searchable iff it has a point and is omniscient:

$$\text{searchable}(X) \iff X \times \text{omniscient}(X).$$

A few theorems rely on pointedness, using the notion of searchability.

Closure under Σ

If X is omniscient/searchable and Y is an X-indexed family of omniscient/searchable types, then so is its disjoint sum $\Sigma(x : X), Y(x)$.

Closure under Π

Not to be expected in general.

E.g. $\mathbb{N}_\infty$ and 2 are omniscient, but in continuous and effective models of type theory, the function space $\mathbb{N}_\infty \to 2$ is not.

In the topological topos, $\mathbb{N}_\infty \to 2$ is a countable discrete space.

Closure under finite products

Theorem A product of searchable types indexed by a finite type is searchable.

Brouwerian closure under countable products

Theorem Brouwerian intuitionistic axioms $\implies$
A countable product of searchable types is searchable.

This is a kind of Tychonoff theorem, if we think of searchability as a "synthetic" notion of compactness.

In particular, the Cantor type $2^{\mathbb{N}}$, which is interpreted as the Cantor space in the topological topos, is searchable.

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   (the effective topos, which is realizability over Kleene's K_1).

2. But validated in another effective model
   (realizability over Kleene's K_2),
   and in the topological topos.

   (I implemented this in Agda, by disabling the termination checker in a particular function. One can run interesting examples.)

We will need this form of closure under Π

Theorem (micro Tychonoff)
A product of searchable types indexed by a subsingleton type is itself searchable.

That is, if X is a subsingleton, and Y is an X-indexed family of searchable types, then the type $\Pi(x : X), Y(x)$ is searchable.

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This cannot be proved if searchability is replaced by omniscience (that is, if we don't assume that every $Y(x)$ is pointed).

This is easy with excluded middle, but we are not assuming it.

Theorem A subsingleton-indexed product of searchable types is searchable.

1. Let X subsingleton, $Y(x)$ searchable for every $x : X$.
2. $Z \stackrel{\text{def}}{=} \Pi(x : X), Y(x)$.

   We have $\Pi(x : X), (Z \simeq Y(x))$ and $(X \to 0) \to (Z \simeq 1)$.

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3. Let $p : Z \to 2$.
4. Construct $z_0(x) \stackrel{\text{def}}{=} \dots$ in Z using the first equivalence.
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   $p(z_0) = 1 \to \Pi(z : Z), p(z) = 1$. Q.E.D.

Amusing consequence, tangential to our development

Consider the subsingleton version of LPO discussed above.

Corollary. *The type* $\mathbb{N}^{LPO}$ *is searchable.*

- The reason is that LPO implies that $\mathbb{N}$ is searchable, and so this is a product of searchable types.

  Even though the searchability of $\mathbb{N}$ is undecided!

- If LPO holds, the type of the corollary is $\mathbb{N}$.
- If LPO fails, it is the contractible type 1.
- As LPO is undecided, we don't know what the type $\mathbb{N}^{\mathsf{LPO}}$ "really is".
- Whatever it is, however, it is always searchable.

Disjoint sum with a point at infinity

Theorem
The disjoint sum of a countable family of searchable sets with a point at infinity is searchable.

We need to say how we add a point at infinity.

The type $1 + \Sigma(n : \mathbb{N}), X(n)$ won't do, of course.

We will do this in a couple of steps.

Injectivity of the universe of types

Theorem
For any embedding $e : A \to B$, every $X : A \to U$ extends to some $Y : B \to U$ along e, up to equivalence,

$$\Pi(a : A), (Y(e(a)) \simeq X(a)).$$

A map $e : A \to B$ is called an embedding iff its fibers $e^{-1}(b)$,

$$\Sigma(a : A), f(a) = b,$$

are all subsingletons.

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Two constructions:

1. We have the "maximal" extension $Y = X/e$.

$$\begin{aligned} (X/e)(b) &= \Pi\left(s : e^{-1}(b)\right), X(\mathrm{pr}_1\, s) \\ &\simeq \Pi(a : A),\ e(a) = b \ \to\ X(a). \end{aligned}$$

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2. And also the "minimal" extension $Y = X \setminus e$.

$$\begin{aligned}(X \setminus e)(b) &= \Sigma\left(s : e^{-1}(b)\right), X(\mathrm{pr}_1\, s) \\ &\simeq \Sigma(a : A),\ e(a) = b \ \times\ X(a).\end{aligned}$$

The first one works our purposes.

Injectivity of the universe of types

Let $e : A \to B$ be an embedding and $X : A \to U$.

Consider the extended type family $X/e : B \to U$ defined above:

$$(X \setminus e)(b) = \Pi \left(s : e^{-1}(b)\right), X(\mathrm{pr}_1\, s)$$

We have

1. For all $b : B$ *not* in the image of the embedding,

$$(X/e)(b) \simeq 1.$$

2. If for all $a : A$, the type $X(a)$ is searchable too, then for all $b : B$ the type $(X/e)(b)$ is searchable, by micro-Tychonoff.

3. Hence if additionally B is searchable, the type $\Sigma(b : B), (X/e)(b)$ is searchable too.

4. We are interested in $A = \mathbb{N}$ and $B = \mathbb{N}_\infty$, which gives the disjoint sum of $X(a)$ with a point at infinity.

A map $L : (\mathbb{N} \to U) \to U$

Let $e : \mathbb{N} \to \mathbb{N}_\infty$ be the natural embedding.

Given $X : \mathbb{N} \to U$, first take $X/e : \mathbb{N}_\infty \to U$

This step adds a point at infinity to the sequence.

We then sum over $\mathbb{N}_\infty$, to get $L(X)$:

$$L(X) = \Sigma(u : \mathbb{N}_\infty), (X/e)(u).$$

Then L maps any sequence of searchable types to a searchable type.

Iterating this map $L : (\mathbb{N} \to U) \to U$

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1. Extensional means that any two elements with the same predecessors are equal.

2. The accessibility of points of X is inductively defined.

   We say that $x : X$ is accessible whenever every $y < x$ is accessible.

   The accessibility of a point is a subingleton.

3. $<$ is accessible if every $x : X$ is accessible.

   The accessibility of $<$ implies that it is subsingleton valued, and that X is set.

The delay monad

Define $F(X) = L(\lambda n.X)$, which is equivalent to $\Sigma(u : \mathbb{N}_\infty), \Pi(n : \mathbb{N}), X^{e(n)=u}$.

An equivalent conindductive definition of F is given by constructors

$$\begin{aligned} \text{now} &: X \to F(X), \\ \text{later} &: F(X) \to F(X). \end{aligned}$$

1. The Cantor type $2^{\mathbb{N}}$ is the carrier of a final coalgebra of F.
2. There is an initial algebra, whose carrier is the subset of Cantor consisting of the sequences with finitely many zeros, for a suitable notion of finiteness.

(Which is classically equivalent to the classical one.)

For the sake of completeness, we characterize the injectives in UF

We have seen that universes are injective, and applied this to construct searchable types and ordinals.

Independently of this, it is natural to try to grasp the injective types.

1. In topos theory, the injectives are the retracts powers of the subobject classifier.

2. We show that, in UF, they are the retracts of powers of universes.

3. Before concluding, we prove this and offer a finer analysis.

The Yoneda embedding

1. For any type X, point $x : X$ and family $A : X \to U$,

$$(\Pi(y : X).\, \mathrm{Id}\; x\, y \to A(y)) \simeq A(x).$$

   This is the Yoneda Lemma.

2. Say that A is representable if we have $x : X$ with $A(y) \simeq \mathrm{Id}\; x\, y$.

3. A having a universal element amounts to $\Sigma(x : X).A(x)$ being contractible.

4. The representability of A is equivalent to the contractibility of $\Sigma(x : X).A(x)$, and hence representability is a proposition.

Therefore, assuming univalence,

Theorem. *The map* $\mathrm{Id} : X \to U^X$ *is an embedding.*

Standard reasoning with injectives

1. Any power I^X of an injective type I is again injective.

   This argument needs that if $A \to B$ is an embedding then so is its product $A \times X \to B \times X$ with the identity map.

2. A retract of an injective type is again injective.

3. An injective type is a retract of every type in which it is embedded.

Characterization of the injective types

Combining this with the Yoneda Embedding:

Theorem. *The injective types are precisely the retracts of powers of the universes.*

We also have:

Theorem. *The injective sets are precisely the retracts of powers of the universe of propositions.*

Theorem. *The injective $n+1$-types are precisely the retracts of powers of the universe of n-types.*

End