String Diagrams *for* Free Monads

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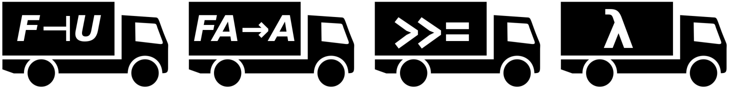
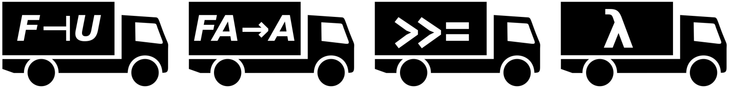


Abstract nonsense → Practical programming



Abstract nonsense → Practical programming

We want our reasoning principles to be:

- **equational:** proofs are by chains of equalities between programs
- **expressive:** leveraging high-level properties of particular constructs

free monads

A type of trees with nodes shaped by the functor `f` and leaves given by variables of the type `a`.

```
data Free f a
  = Var a
  | Op (f (Free f a))
```

Monadic bind given by substitution in leaves.

Sometimes denoted F^*.

the problem

- Inductive data types (initial algebra) come with a low-level resoning principle (structural induction).

- However, structural induction can be a bit awkward.

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- Inductive data types (initial algebra) come with a low-level resoning principle (structural induction).

- However, structural induction can be a bit awkward.

Consider the **free monad transformer**:

```
newtype FreeT f m a = FreeT (m (Free (f . m) a)
```

Would **you** like to prove the monad laws using structural induction by hand?

Below, we prove that $m \cdot \mu^K_K = M(\mathsf{id}+\Sigma\mu^K_K)\cdot w$ and $m \cdot K\mu^K = M(\mathsf{id}+\Sigma K\mu^K)\cdot w$, which means that both μ^K_K and $K\mu^K$ are coalgebra homomorphisms $\mu^K_K, K\mu^K : \langle K^3, w\rangle \to \langle K^2, m\rangle$. By uniqueness, $\mu^K \cdot \mu^K_K = [\![m]\!] \cdot \mu^K_K = [\![w]\!] = [\![m]\!] \cdot K\mu^K = \mu^K \cdot K\mu^K$. Diagrammatically:

$m \cdot \mu^K_K$

$=$ { definition of m, universal property of μ^K }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{id}+\Sigma K,\Sigma K^2}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2}) \cdot \alpha_K \cdot \alpha_K^{-1} \cdot M(\mathsf{id}_K+\mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}_K+\mathsf{id}_{\Sigma K^2}+\Sigma\mu^K_K)$
$\cdot \mathsf{flat}_{K+\Sigma K^2,\Sigma K^3} \cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2}$

$=$ { cancellation of α_K and α_K^{-1}, and naturality of α }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{id}+\Sigma K,\Sigma K^2}$
$\cdot M(\mathsf{id}_{M(\mathsf{id}+\Sigma K)}+\mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}_{M(\mathsf{id}+\Sigma K)}+\mathsf{id}_{\Sigma K^2}+\Sigma\mu^K_K)$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2+\Sigma K^3}) \cdot \mathsf{flat}_{K+\Sigma K^2,\Sigma K^3} \cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2}$

$=$ { naturality of flat }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2}) \cdot M(\mathsf{id}_{\mathsf{id}+\Sigma K}+\mathsf{gr}_{\Sigma K^2})$
$\cdot M(\mathsf{id}_{\mathsf{id}+\Sigma K}+\mathsf{id}_{\Sigma K^2}+\Sigma\mu^K_K) \cdot \mathsf{flat}_{\mathsf{id}+\Sigma K,\Sigma K^2+\Sigma K^3}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2+\Sigma K^3}) \cdot \mathsf{flat}_{K+\Sigma K^2,\Sigma K^3} \cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2}$

$=$ { distributivity of composition over coporduct }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}_{\mathsf{id}+\Sigma K^2}+\mathsf{gr}_{\Sigma K^2})$
$\cdot M(\mathsf{id}_{\mathsf{id}}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2}+\Sigma\mu^K_K) \cdot \mathsf{flat}_{\mathsf{id}+\Sigma K,\Sigma K^2+\Sigma K^3}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2+\Sigma K^3}) \cdot \mathsf{flat}_{K+\Sigma K^2,\Sigma K^3} \cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2}$

$=$ { naturality of gr, right unit for K }

$M(\mathsf{id}+\Sigma\mu^K_K) \cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K^3}) \cdot M(\mathsf{id}_{\mathsf{id}+\Sigma K^3}+\mathsf{gr}_{\Sigma K^3})$
$\cdot M(\mathsf{id}_{\mathsf{id}}+\Sigma(\eta^K_{K^2}\cdot\eta^K_K)+\Sigma\eta^K_{K^2}+\Sigma\mathsf{id}_{K^3}) \cdot \mathsf{flat}_{\mathsf{id}+\Sigma K,\Sigma K^2+\Sigma K^3}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2+\Sigma K^3}) \cdot \mathsf{flat}_{K+\Sigma K^2,\Sigma K^3} \cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2}$

$=$ { definition of w }

$M(\mathsf{id}+\Sigma\mu^K_K) \cdot w$

* * *

$m \cdot K\mu^K$

$=$ { definition of m }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{id}+\Sigma K,\Sigma K^2}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2}) \cdot \alpha_K \cdot K\mu^K$

$=$ { naturality of α }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{id}+\Sigma K,\Sigma K^2}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2}) \cdot M(\mu^K+\Sigma K\mu^K) \cdot \alpha_{K^2}$

$=$ { properties of coproducts }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{id}+\Sigma K,\Sigma K^2}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2}) \cdot M(\mathsf{id}_K+\Sigma K\mu^K) \cdot M(\mu^K+\mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2}$

$=$ { universal property of μ^K }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{id}+\Sigma K,\Sigma K^2}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2}) \cdot M(\mathsf{id}_K+\Sigma K\mu^K) \cdot M(\alpha^{-1}+\mathsf{id}_{\Sigma K^3})$
$\cdot M(M(\mathsf{id}+\mathsf{gr}_{\Sigma K})+\mathsf{id}_{\Sigma K^3}) \cdot M(M(\mathsf{id}+\mathsf{id}_{\Sigma K}+\Sigma\mu^K)+\mathsf{id}_{\Sigma K^3})$
$\cdot M(\mathsf{flat}_{\mathsf{id}+\Sigma K^2,\Sigma K^2}+\mathsf{id}_{\Sigma K^3}) \cdot M(M(\alpha+\mathsf{id}_{\Sigma K^2})+\mathsf{id}_{\Sigma K^3})$
$\cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2}$

$=$ { cancellation of α and α^{-1} }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{id}+\Sigma K,\Sigma K^2}$
$\cdot M(\mathsf{id}_{M(\mathsf{id}+\Sigma K)}+\Sigma K\mu^K)$
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$=$ { naturality of flat }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2}) \cdot M(\mathsf{id}_{\mathsf{id}+\Sigma K}+\Sigma K\mu^K)$
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$=$ { monad laws }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2}) \cdot M(\mathsf{id}_{\mathsf{id}+\Sigma K}$ …
$\cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K}+\mathsf{id}_{\Sigma K^3}) \cdot M(\mathsf{id}+\mathsf{id}_{\Sigma K}+\Sigma\mu^K+$ …
$\cdot \mathsf{flat}_{\mathsf{id}+\Sigma K,\Sigma K^2+\Sigma K^3} \cdot M(\alpha+\mathsf{id}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^3})$
$\cdot \mathsf{flat}_{K+\Sigma K^2,\Sigma K^3} \cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2}$

$=$ { naturality of gr }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^2})$
$\cdot M(\mathsf{id}+\Sigma\eta^K_K+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2}) \cdot M(\mathsf{id}_{\mathsf{id}+\Sigma K+\Sigma K}+$ …
$\cdot M(\mathsf{id}+\mathsf{id}_{\Sigma K}+\Sigma\mu^K+\mathsf{id}_{\Sigma K^3}) \cdot \mathsf{flat}_{\mathsf{id}+\Sigma K,\Sigma K^2+\Sigma}$ …
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^3}) \cdot \mathsf{flat}_{K+\Sigma K^2,\Sigma K^3} \cdot M(\alpha_K$ …

$=$ { properties of coproducts }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^2})$
$\cdot M(\mathsf{id}+\Sigma\eta^K_K+\Sigma(\eta^K_K\cdot\mu^K)+\Sigma K\mu^K) \cdot \mathsf{flat}_{\mathsf{id}+\Sigma K}$ …
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^3}) \cdot \mathsf{flat}_{K+\Sigma K^2,\Sigma K^3} \cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2}$

$=$ { naturality of η^K }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^2})$
$\cdot M(\mathsf{id}+\Sigma\eta^K_K+\Sigma(K\mu^K\cdot\eta^K_{K^2})+\Sigma K\mu^K) \cdot \mathsf{flat}_{\mathsf{id}+\Sigma K,\Sigma K^2+\Sigma K^3}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^3}) \cdot \mathsf{flat}_{K+\Sigma K^2,\Sigma K^3} \cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2}$

$=$ { right unit for K, naturality of gr }

$M(\mathsf{id}+\Sigma K\mu^K) \cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K^3}) \cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K^3}+\mathsf{id}_{\Sigma K^3})$
$\cdot M(\mathsf{id}+\Sigma(\eta^K_{K^2}\cdot\eta^K_K)+\Sigma\eta^K_{K^2}+\mathsf{id}_{\Sigma K^3}) \cdot \mathsf{flat}_{\mathsf{id}+\Sigma K,\Sigma K^2+\Sigma K^3}$
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$=$ { definition of w }

$M(\mathsf{id}+\Sigma K\mu^K) \cdot w$ □

(From an unpublished appendix to M.Piróg & J.Gibbons. *Monads for behaviour.* MFPS 2013)

the problem

- Let's invent more abstract reasoning principles that respect the underlying structure!

string **diagrams**

- A 2-dimensional notation for expressions
- Useful for presentation of more complicated expressions that involve a number of functors
- Some administrative equalities are **built-in**

lookup

Consider the function

$$\texttt{lookup k :: } \forall \texttt{a.List (Key, a)} \rightarrow \texttt{Maybe a}$$

It is polymorphic in `a`. So, alternatively, we write

$$\text{lookup k} : \text{List} \circ \text{(Key, -)} \rightarrow \text{Maybe}$$

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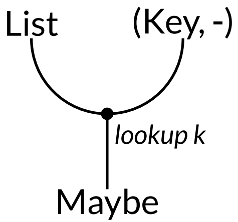
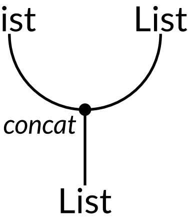
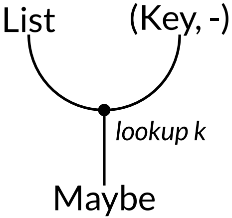


composition





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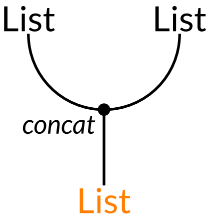

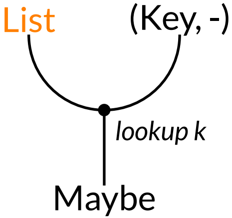

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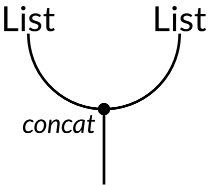

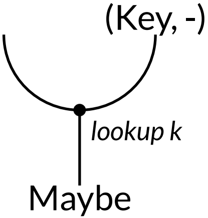

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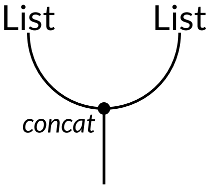

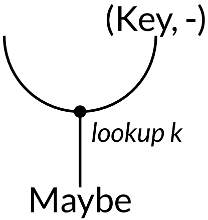

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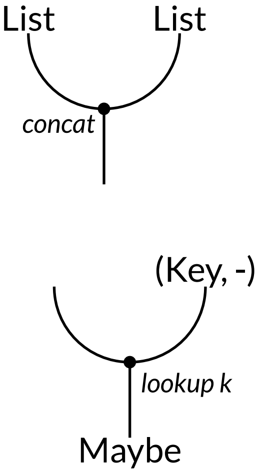

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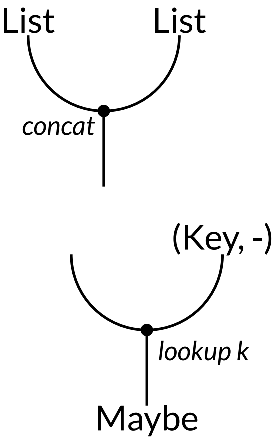

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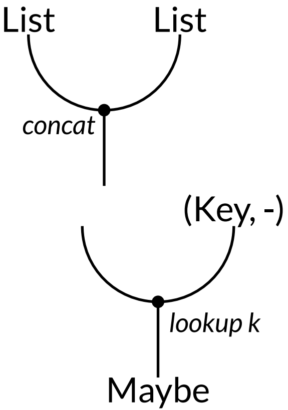

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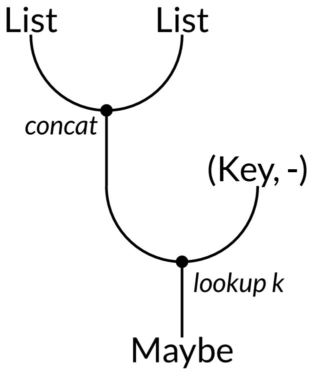
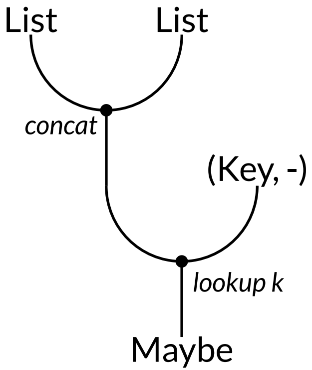


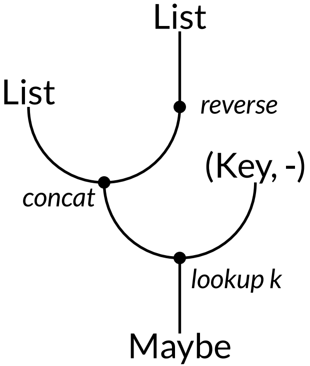
composition

`lookup k . concat ::` $\forall$`a.List (List (Key, a))` $\to$ `Maybe a`



composition

```
lookup k . concat . fmap reverse
:: ∀a.List (List (Key, a)) → Maybe a
```



free monads, more abstractly

This definition is only one of many possible implementations:

```
data Free f a
  = Var a
  | Op (f (Free f a))
```

Shouldn't this be an interface?

```
class FreeMonad f a where ???
```

What are the **operations** and **laws** that characterise free monads?

freeness, mathematically

```
emb :: ∀a.f a → Free f a
```

$$\text{emb} : F \to F^*$$

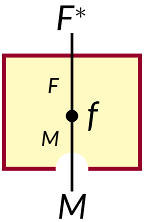
Generic operation that takes a single operation to a term.

freeness, mathematically

```
interp :: (Functor f, Monad m) ⇒
    (∀x.f x -> m x) -> Free f a → m a
```

Given a monad M and $f : F \to M$,
there is a **monad morphism**

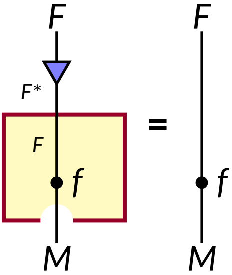
$$\lfloor f \rfloor : F^* \to M$$



cancellation

```
interp f . emb = f
```

$$\lfloor f \rfloor \cdot \mathsf{emb} = f$$

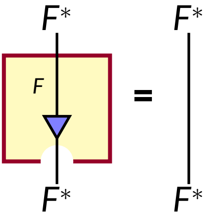


Interpreting a term that consists of one operation is the same as interpreting that operation.

reflection

```
interp emb = id
```

$$\lfloor \text{emb} \rfloor = \text{id}$$

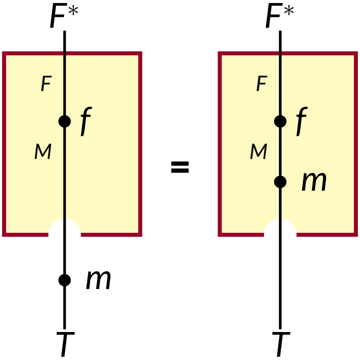


Interpreting operations as syntax... preserves syntax.

fusion

```
m . interp f = interp (m . f)
```

$m \cdot \lfloor f \rfloor = \lfloor m \cdot f \rfloor$, where m is a monad morphism



Interpreting a term and then transforming semantics is the same as interpreting and transforming the term in one go.

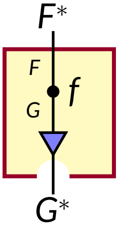
example: **renaming is functorial**

Given functors F and G, and a natural transformation $f : F \to G$, we can define a natural transformation (*hoist*)

$$F^* \to G^*$$

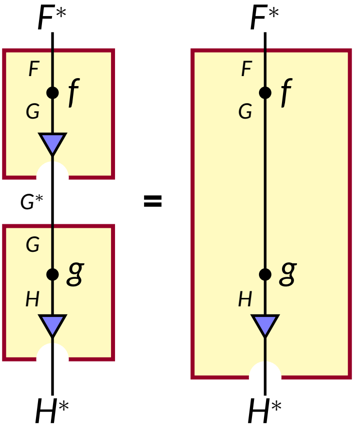
as follows:

$$\lfloor \mathrm{emb} \cdot f \rfloor$$

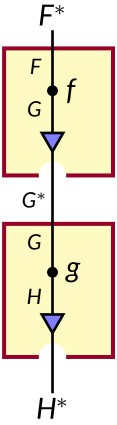


Given three functors F, G, H, and two natural transformations $f : F \to G$ and $g : G \to H$, is the composition of hoists a hoist of composition? That is:

$$\lfloor \mathrm{emb} \cdot g \rfloor \cdot \lfloor \mathrm{emb} \cdot f \rfloor = \lfloor \mathrm{emb} \cdot g \cdot f \rfloor$$



example: **renaming is functorial**



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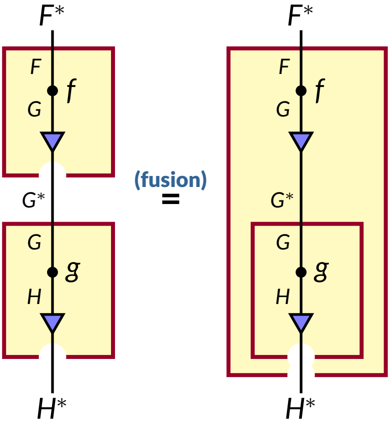
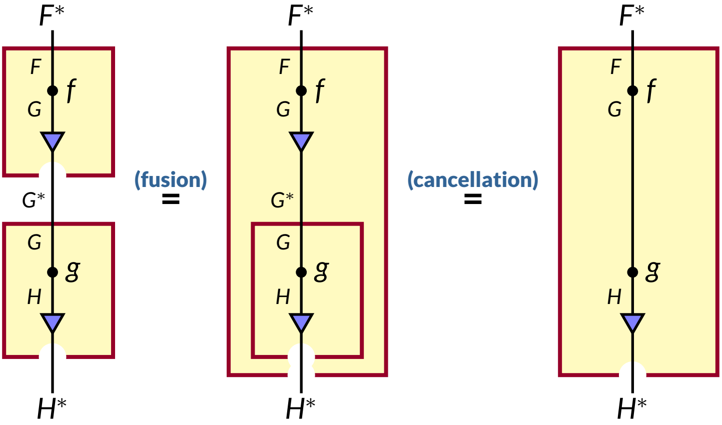


example: **renaming is functorial**



going back to FreeT

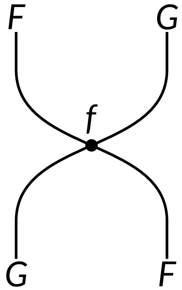
Sadly, what we have seen is not enough to show that **FreeT** is a monad.

Luckily, free monads in Haskell have other kinds of similar properties.

distributable free monads

Given

$f : F \circ G \to G \circ F$

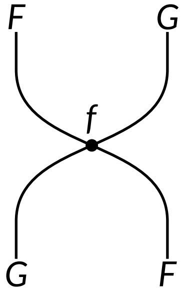


distributable free monads

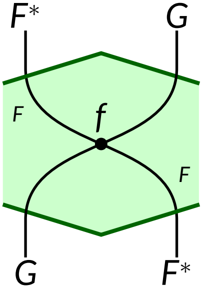
Given

$f : F \circ G \to G \circ F$

$\langle f \rangle : F^* \circ G \to G \circ F^*$

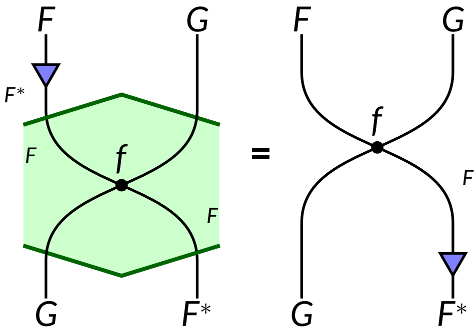


we get



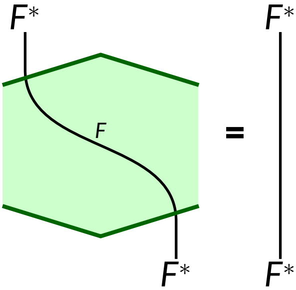
cancellation

$$\langle f \rangle \cdot \mathsf{emb} = G\,\mathsf{emb} \cdot f$$



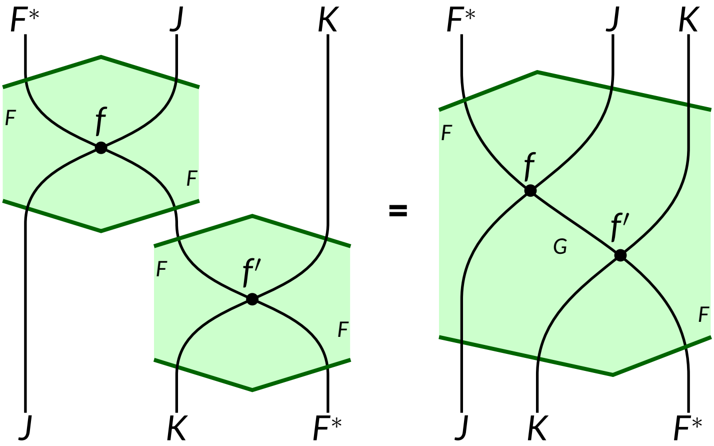
reflection

$$\langle \mathrm{id} \rangle = \mathrm{id}$$



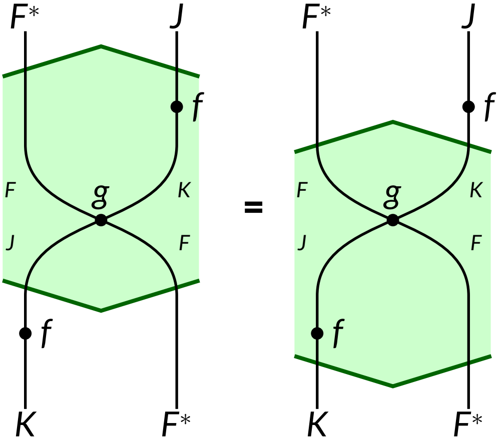
fusion

$$J\langle f' \rangle \cdot \langle f \rangle K = \langle Jf' \cdot fK \rangle$$



uniformity

$$fF^* \cdot \langle g \cdot F^* f \rangle = \langle fF \cdot g \rangle \cdot F^* f$$



uniformity

Kind of

$$(fg)^*f = f(gf)^*$$

in Kleene algebra

Below, we prove that $m\cdot\mu^K_K = M(\mathsf{id}+\Sigma\mu^K_K)\cdot w$ and $m\cdot K\mu^K = M(\mathsf{id}+\Sigma K\mu^K)\cdot w$, which means that both μ^K_K and $K\mu^K$ are coalgebra homomorphisms $\mu^K_K, K\mu^K : \langle K^3, w\rangle \to \langle K^2, m\rangle$. By uniqueness, $\mu^K\cdot\mu^K_K = [\![m]\!]\cdot\mu^K_K = [\![w]\!] = [\![m]\!]\cdot K\mu^K = \mu^K\cdot K\mu^K$. Diagrammatically:

$m\cdot\mu^K_K$

$=$ { definition of m, universal property of μ^K }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2})\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2})\cdot\alpha_K\cdot\alpha_K^{-1}\cdot M(\mathsf{id}_K+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}_K+\mathsf{id}_{\Sigma K^2}+\Sigma\mu^K_K)$
$\cdot\mathsf{flat}_{K+\Sigma K^2,\Sigma K^3}\cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3})\cdot\alpha_{K^2}$

$=$ { cancellation of α_K and α_K^{-1}, and naturality of α }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2})\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2}$
$\cdot M(\mathsf{id}_{M(\mathsf{Id}+\Sigma K)}+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}_{M(\mathsf{Id}+\Sigma K)}+\mathsf{id}_{\Sigma K^2}+\Sigma\mu^K_K)$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2+\Sigma K^3})\cdot\mathsf{flat}_{K+\Sigma K^2,\Sigma K^3}\cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3})\cdot\alpha_{K^2}$

$=$ { naturality of flat }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2})\cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K}+\mathsf{gr}_{\Sigma K^2})$
$\cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K}+\mathsf{id}_{\Sigma K^2}+\Sigma\mu^K_K)\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2+\Sigma K^3}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2+\Sigma K^3})\cdot\mathsf{flat}_{K+\Sigma K^2,\Sigma K^3}\cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3})\cdot\alpha_{K^2}$

$=$ { distributivity of composition over coporduct }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K^2}+\mathsf{gr}_{\Sigma K^2})$
$\cdot M(\mathsf{id}_{\mathsf{Id}}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2}+\Sigma\mu^K_K)\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2+\Sigma K^3}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2+\Sigma K^3})\cdot\mathsf{flat}_{K+\Sigma K^2,\Sigma K^3}\cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3})\cdot\alpha_{K^2}$

$=$ { naturality of gr, right unit for K }

$M(\mathsf{id}+\Sigma\mu^K_K)\cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K^3})\cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K^3}+\mathsf{gr}_{\Sigma K^3})$
$\cdot M(\mathsf{id}_{\mathsf{Id}}+\Sigma(\eta^K_{K^2}\cdot\eta^K_K)+\Sigma\eta^K_{K^2}+\Sigma\mathsf{id}_{K^3})\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2+\Sigma K^3}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2+\Sigma K^3})\cdot\mathsf{flat}_{K+\Sigma K^2,\Sigma K^3}\cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3})\cdot\alpha_{K^2}$

$=$ { definition of w }

$M(\mathsf{id}+\Sigma\mu^K_K)\cdot w$

* * *

$m\cdot K\mu^K$

$=$ { definition of m }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2})\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2})\cdot\alpha_K\cdot K\mu^K$

$=$ { naturality of α }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2})\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2})\cdot M(\mu^K+\Sigma K\mu^K)\cdot\alpha_{K^2}$

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$=$ { properties of coproducts }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2})\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2})\cdot M(\mathsf{id}_K+\Sigma K\mu^K)\cdot M(\mu^K+\mathsf{id}_{\Sigma K^3})\cdot\alpha_{K^2}$

$=$ { universal property of μ^K }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2})\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2})\cdot M(\mathsf{id}_K+\Sigma K\mu^K)\cdot M(\alpha^{-1}+\mathsf{id}_{\Sigma K^3})$
$\cdot M(M(\mathsf{id}+\mathsf{gr}_{\Sigma K})+\mathsf{id}_{\Sigma K^3})\cdot M(M(\mathsf{id}+\mathsf{id}_{\Sigma K}+\Sigma\mu^K)+\mathsf{id}_{\Sigma K^3})$
$\cdot M(\mathsf{flat}_{\mathsf{Id}+\Sigma K^2,\Sigma K^2}+\mathsf{id}_{\Sigma K^3})\cdot M(M(\alpha+\mathsf{id}_{\Sigma K^2})+\mathsf{id}_{\Sigma K^3})$
$\cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3})\cdot\alpha_{K^2}$

$=$ { cancellation of α and α^{-1} }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2})\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2}$
$\cdot M(\mathsf{id}_{M(\mathsf{Id}+\Sigma K)}+\Sigma K\mu^K)$
$\cdot M(M(\mathsf{id}+\mathsf{gr}_{\Sigma K})+\mathsf{id}_{\Sigma K^3})\cdot M(M(\mathsf{id}+\mathsf{id}_{\Sigma K}+\Sigma\mu^K)+\mathsf{id}_{\Sigma K^3})$
$\cdot M(\mathsf{flat}_{\mathsf{Id}+\Sigma K^2,\Sigma K^2}+\mathsf{id}_{\Sigma K^3})\cdot M(M(\alpha+\mathsf{id}_{\Sigma K^2})+\mathsf{id}_{\Sigma K^3})$
$\cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3})\cdot\alpha_{K^2}$

$=$ { naturality of flat }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2})\cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K}+\Sigma K\mu^K)$
$\cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K}+\mathsf{id}_{\Sigma K^3})\cdot M(\mathsf{id}+\mathsf{id}_{\Sigma K}+\Sigma\mu^K+\mathsf{id}_{\Sigma K^3})$
$\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K^2+\Sigma K^2,\Sigma K^3}\cdot M(\mathsf{flat}_{\mathsf{Id}+\Sigma K^2,\Sigma K^2}+\mathsf{id}_{\Sigma K^3})$
$\cdot M(M(\alpha+\mathsf{id}_{\Sigma K^2})+\mathsf{id}_{\Sigma K^3})\cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3})\cdot\alpha_{K^2}$

$=$ { monad laws }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2})\cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K}$ …
$\cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K}+\mathsf{id}_{\Sigma K^3})\cdot M(\mathsf{id}+\mathsf{id}_{\Sigma K}+\Sigma\mu^K+\mathsf{i}$ …
$\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2+\Sigma K^3}\cdot M(\alpha+\mathsf{id}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^3})$
$\cdot\mathsf{flat}_{K+\Sigma K^2,\Sigma K^3}\cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3})\cdot\alpha_{K^2}$

$=$ { naturality of gr }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^2})$
$\cdot M(\mathsf{id}+\Sigma\eta^K_K+\Sigma\eta^K_K+\mathsf{id}_{\Sigma K^2})\cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K+\Sigma K}+$ …
$\cdot M(\mathsf{id}+\mathsf{id}_{\Sigma K}+\Sigma\mu^K+\mathsf{id}_{\Sigma K^3})\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2+\Sigma}$ …
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^3})\cdot\mathsf{flat}_{K+\Sigma K^2,\Sigma K^3}\cdot M(\alpha_K$ …

$=$ { properties of coproducts }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^2})$
$\cdot M(\mathsf{id}+\Sigma\eta^K_K+\Sigma(\eta^K_K\cdot\mu^K)+\Sigma K\mu^K)\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K}$ …
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^3})\cdot\mathsf{flat}_{K+\Sigma K^2,\Sigma K^3}\cdot M(\alpha_K+$ …

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$=$ { naturality of η^K }

$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^2})$
$\cdot M(\mathsf{id}+\Sigma\eta^K_K+\Sigma(K\mu^K\cdot\eta^K_{K^2})+\Sigma K\mu^K)\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2+\Sigma K^3}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^3})\cdot\mathsf{flat}_{K+\Sigma K^2,\Sigma K^3}\cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3})\cdot\alpha_{K^2}$

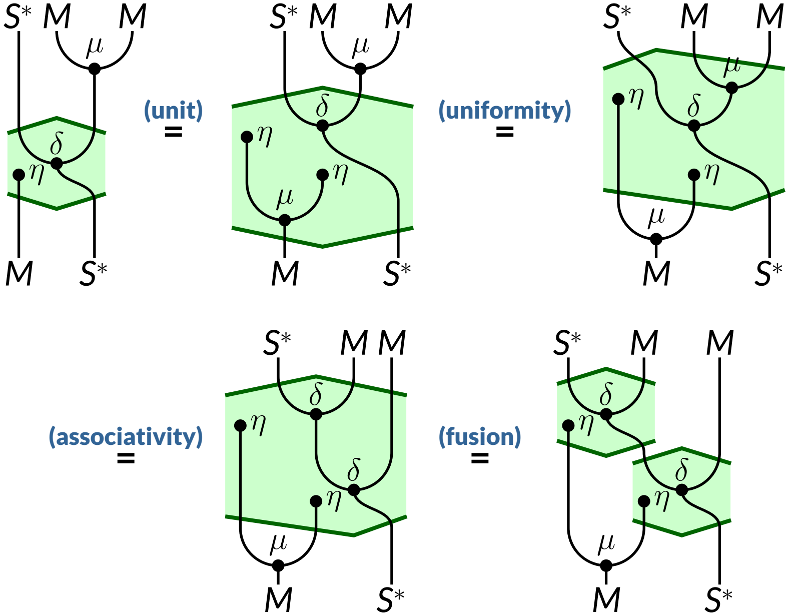
$=$ { right unit for K, naturality of gr }

$M(\mathsf{id}+\Sigma K\mu^K)\cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K^3})\cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K^3}+\mathsf{id}_{\Sigma K^3})$
$\cdot M(\mathsf{id}+\Sigma(\eta^K_{K^2}\cdot\eta^K_K)+\Sigma\eta^K_{K^2}+\mathsf{id}_{\Sigma K^3})\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2+\Sigma K^3}$
$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^3})\cdot\mathsf{flat}_{K+\Sigma K^2,\Sigma K^3}\cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3})\cdot\alpha_{K^2}$

$=$ { definition of w }

$M(\mathsf{id}+\Sigma K\mu^K)\cdot w$ □

(From an unpublished appendix to M.Piróg & J.Gibbons. *Monads for behaviour.* MFPS 2013)



(unit)
=
(uniformity)
=
(associativity)
=
(fusion)
=

more in the ICFP 2016 paper

- one more universal property (generalised fold)
- relationship between different universal properties
- a lot of further examples and an equational proof of a real-life theorem
- explanation where all these properties come from