Intuitionistic Temporal Logic from Reactive Programming

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Functional reactive programming (FRP)

- extension of functional programming with support for describing temporal behavior
- time is a totally ordered set $(T, \leq)$:
  - linear
  - not necessarily discrete
- key constructs for behavior descriptions:
  - signals
  - events

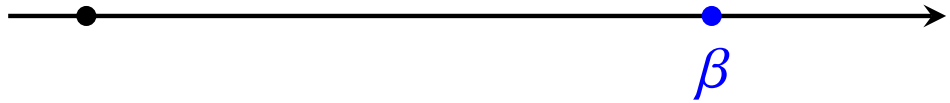
Signals and events

- signals are time-varying values:

$Signal_\alpha$:

- events are times with associated values:

$Event_\beta$:



- examples:

  $Signal_{\mathbb{R}}$ audio channel in a multimedia application

  $Event_\Sigma$ key press

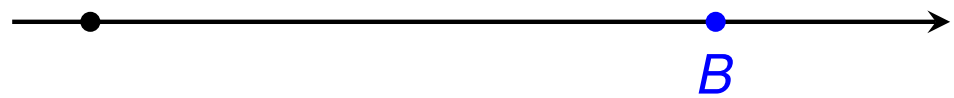
Correspondence to temporal logic operators

- *Signal* corresponds to a future-only variant of $\Box$:

$\Box' A$:

- *Event* corresponds to a future-only variant of $\Diamond$:

$\Diamond' B$:



Establishment of a Curry–Howard correspondence

- FRP inspires temporal logic, making it intuitionistic in several ways:
  - double negation elimination does not hold:

$$\neg\neg A \not\vdash A$$

  - $\Box'$ and $\Diamond'$ are not interdefinable:

$$\Box' A \not\equiv \neg\Diamond'\neg A$$
$$\Diamond' B \not\equiv \neg\Box'\neg B$$

  - $\Diamond'$ does not distribute over finite disjunctions:

$$\Diamond'(A \vee B) \not\vdash \Diamond' A \vee \Diamond' B \qquad \Diamond'\bot \not\vdash \bot$$
$$\Diamond' A \vee \Diamond' B \vdash \Diamond'(A \vee B) \qquad \bot \vdash \Diamond'\bot$$

- temporal logic inspires FRP:

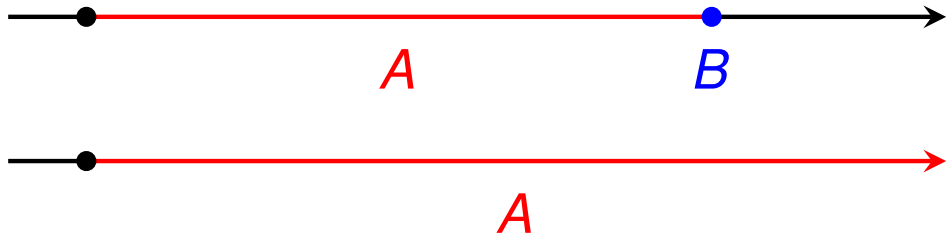
time-dependent truth $\longrightarrow$ time-dependent type inhabitance

"until" proofs $\longrightarrow$ processes

“Until” operators

- family of future-only “until” operators:

$A \rhd_c'' B$:



- parameter c is termination constraint:
  - termination behaviors are elements of $T \cup \{\infty\}$:
    $t \in T$ termination at time t
    ∞ nontermination
  - termination constraints are downward closed sets of permitted termination behaviors
  - special cases:
    T termination guaranteed
    $T \cup \{\infty\}$ nontermination possible
  - shortcuts:

$$\downarrow = T \qquad \updownarrow = T \cup \{\infty\}$$

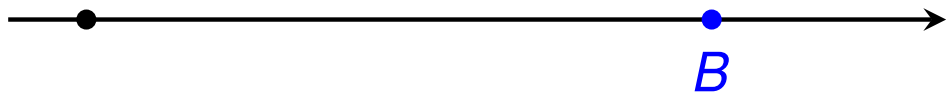
Derivation of “always” and “eventually” operators

- derivation of $\Box'$:

$$\Box' A = A \rhd''_{\updownarrow} \bot:$$

- derivation of $\Diamond'$:

$$\Diamond' B = \top \rhd''_{\downarrow} B:$$



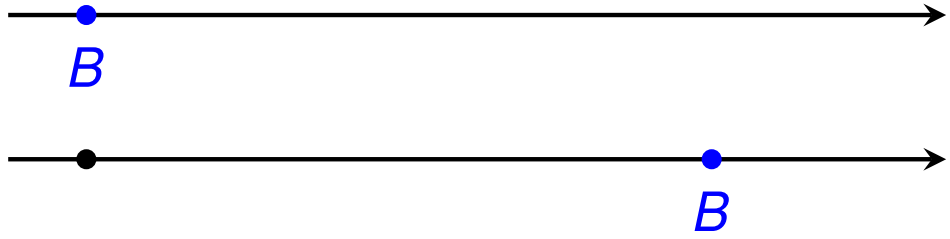
Present-also versions of “always” and “eventually”

- present-also version of $\Box'$:

$\Box A = A \times \Box' A$:

- present-also version of $\Diamond'$:

$\Diamond B = B + \Diamond' B$:



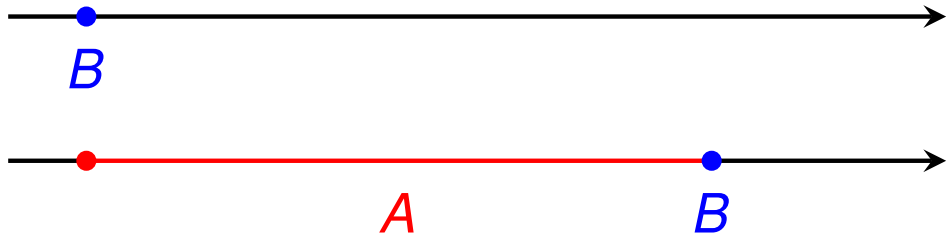
Present-also versions of “until”

- derivation analogous to derivation of $\Box$:

$A \rhd'_c B = A \times A \rhd''_c B$:

- derivation analogous to derivation of $\Diamond$:

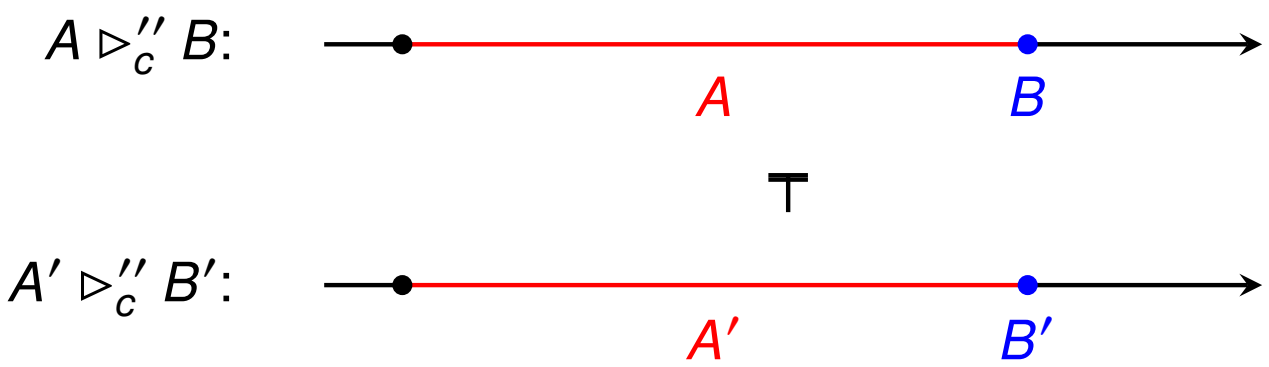
$A \rhd_c B = B + A \rhd'_c B$:



Map

- judgment application under $\triangleright''$:

$$\frac{A \vdash A' \quad B \vdash B'}{A \triangleright''_c B \vdash A' \triangleright''_c B'} \quad \text{(Map)}$$



Join

- elimination of right nesting:

$$\frac{}{A \rhd''_c (A \rhd_c B) \vdash A \rhd''_c B} \quad \text{(Join)}$$

$A \rhd''_c (A \rhd_c B)$:

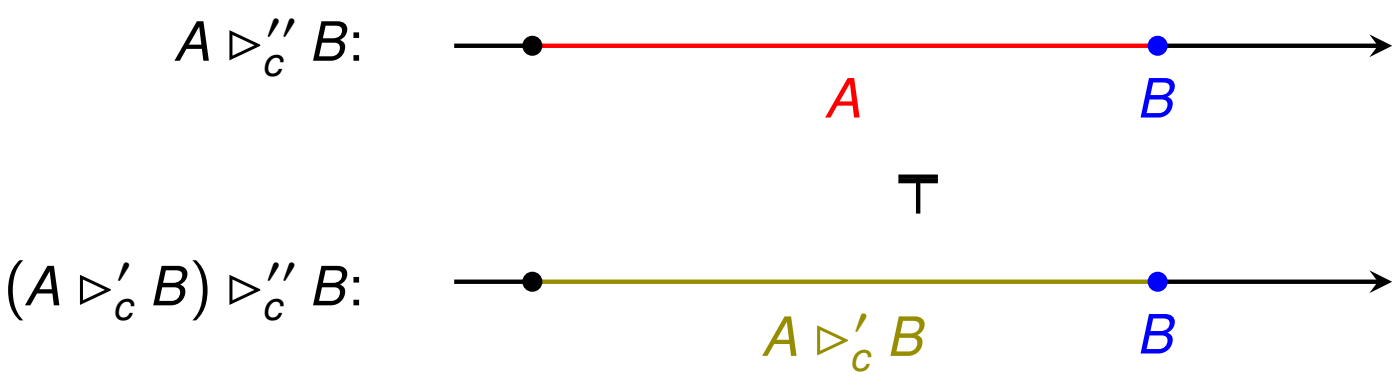
$\top$

$A \rhd''_c B$:

Expand

- introduction of left nesting:

$$\frac{}{A \rhd''_c B \vdash (A \rhd'_c B) \rhd'' B} \quad \text{(Expand)}$$



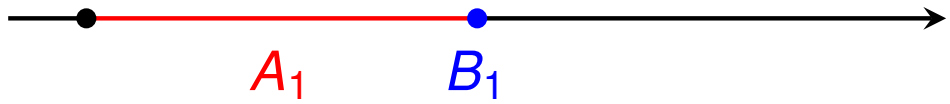
Merge

- merging of two $\rhd''$-statements:

$$\frac{}{A_1 \rhd''_{c_1} B_1 \times A_2 \rhd''_{c_2} B_2 \vdash (A_1 \times A_2) \rhd''_{c_1 \cap c_2} R} \quad \text{(Merge)}$$

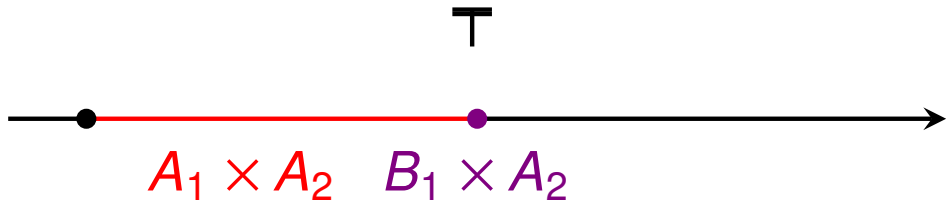
$$R = B_1 \times A_2 + A_1 \times B_2 + B_1 \times B_2$$

$A_1 \rhd''_{c_1} B_1$:



$A_2 \rhd''_{c_2} B_2$:

$(A_1 \times A_2) \rhd''_{c_1 \cap c_2} R$:



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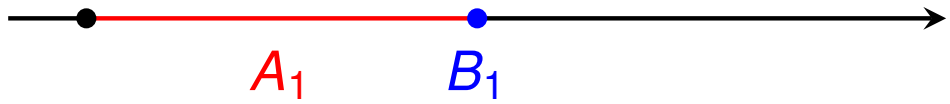

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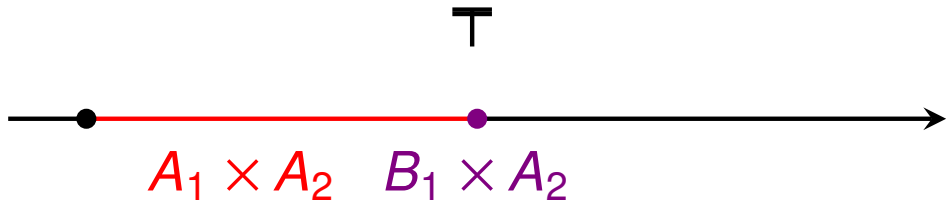

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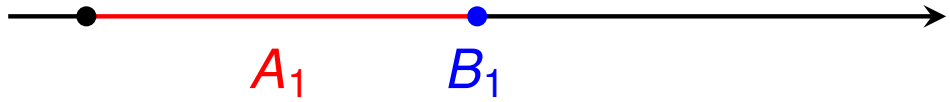

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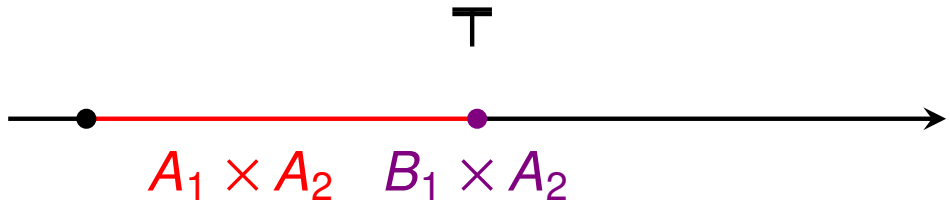

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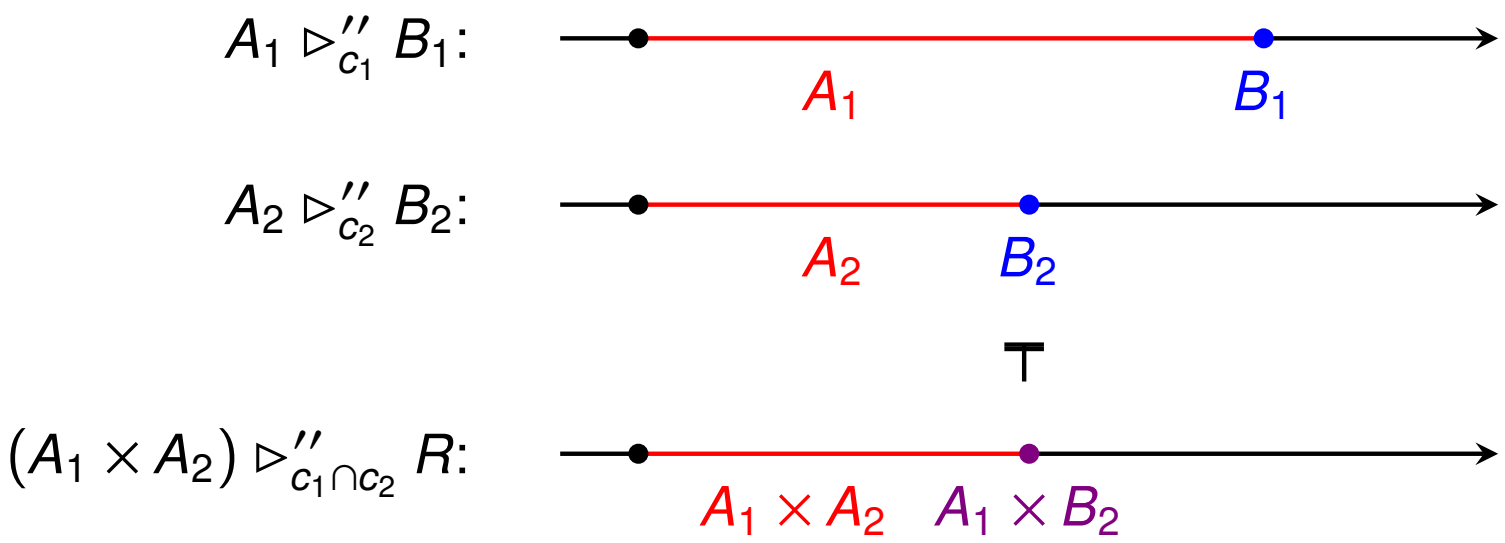

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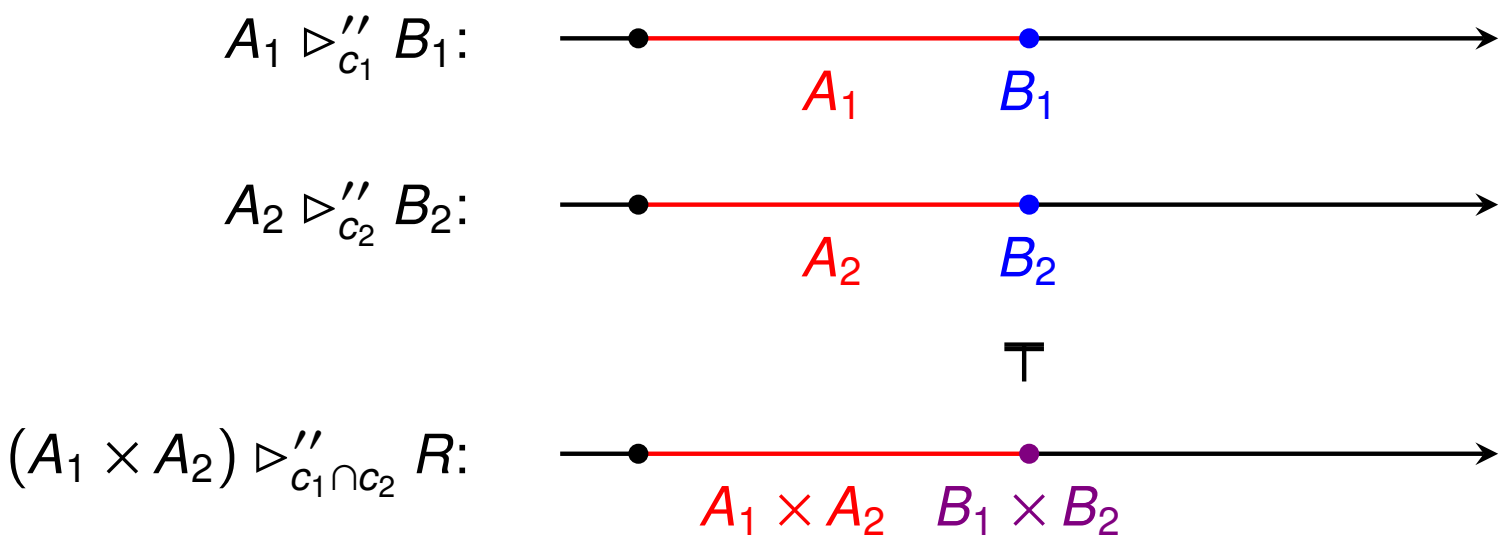

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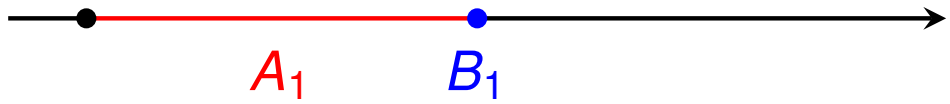
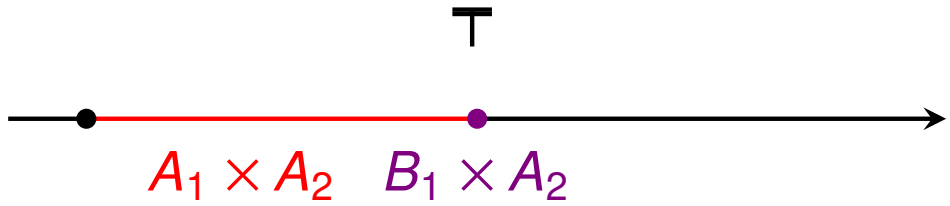


$A_2 \rhd''_{c_2} B_2$:

$(A_1 \times A_2) \rhd''_{c_1 \cap c_2} R$:



Nontermination introduction

- nullary variant of Merge:

$$\frac{}{\top \vdash \top \rhd\!\!\!\rhd^{\prime\prime}_{\updownarrow} \bot}$$ (Nontermination introduction)

$\top \rhd\!\!\!\rhd^{\prime\prime}_{\updownarrow} \bot$:

Conclusions and outlook

- conclusions:
  - discovery of correspondence between FRP and temporal logic profits both FRP and temporal logic
  - causality as a temporal intuitionistic concept
  - inference rules inspired by core operations of FRP
- outlook:
  - integration of our intuitionistic temporal logic and the logic of bunched implication
  - interpretation of the resulting logic as a variant of FRP with support for side effects