Categorical Semantics for an Intuitionistic Temporal Logic

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Temporal logic and functional reactive programming

- temporal logic of this talk:
  - intuitionistic
  - linear notion of time
  - time not necessarily discrete
- Curry–Howard correspondence to functional reactive programming (Jeltsch 2012):
  - programming calculus that also handles temporal aspects of programs
  - time-dependent type membership
  - temporal type constructors

Simple categorical models

- ingredients:

  totally ordered set $(T, \leq)$ time scale

  CCCC $\mathcal{B}$ nontemporal propositions and proofs

- functor category $\mathcal{B}^T$ models temporal propositions and proofs:

$$\begin{array}{ccccccc} \cdots & A(t) & \cdots & A(t') & \cdots \\ & \downarrow{\scriptstyle f_t} & & \downarrow{\scriptstyle f_{t'}} & \\ \cdots & B(t) & \cdots & B(t') & \cdots \end{array}$$

Temporal operators

- future-only strong and weak "until" modeled by functors $\rhd''$ and $\blacktriangleright''$:

$$(A \rhd'' B)(t) = \coprod_{t' \in (t,\infty)} \left(\left(\prod_{t'' \in (t,t')} A(t'') \right) \times B(t') \right)$$

$$(A \blacktriangleright'' B)(t) = (A \rhd'' B)(t) + \prod_{t' \in (t,\infty)} A(t')$$

- variants that also deal with the present:

$$A \rhd' B = A \times A \rhd'' B \qquad A \rhd B = B + A \rhd' B$$

$$A \blacktriangleright' B = A \times A \blacktriangleright'' B \qquad A \blacktriangleright B = B + A \blacktriangleright' B$$

- "always" and "eventually" as special cases:

$$\Box' A = A \blacktriangleright'' 0 \qquad \Box A = A \blacktriangleright' 0$$

$$\Diamond' B = 1 \rhd' B \qquad \Diamond B = 1 \rhd B$$

Topic of this talk

- axiomatically defined categorical semantics for this temporal logic
- road to this semantics:
  1. categorical models of intuitionistic S4
     (Kobayashi 1997; Bierman and de Paiva 2000)
  2. temporal categories for temporal logic with "always" and "eventually"
     (Jeltsch 2012)
  3. abstract process categories for temporal logic with "until"
     (Jeltsch 2014; this talk)

Basic structure

- cartesian closed category $\mathcal{C}$ with coproducts
- functors that model strong and weak "until:"

$$\rhd'', \blacktriangleright'' : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

- unified functor for modeling "until:"

$$- \rhd''_{-} - : \mathbf{2} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

- traditional "until" functors by specialization:

$$\rhd'' = \rhd''_0$$
$$\blacktriangleright'' = \rhd''_1$$

- weakening as mapping:

$$\frac{w : 0 \to 1}{\rhd''_w : \rhd'' \to \blacktriangleright''}$$

Comonads and more

- three kinds of structures:
  - comonads:

$$\varepsilon_{A,W,B} : A \rhd'_W B \to A$$
$$\delta_{A,W,B} : A \rhd'_W B \to \left(A \rhd'_W B\right) \rhd'_W B$$

  - ideal comonads:

$$\delta'_{A,W,B} : A \rhd''_W B \to \left(A \rhd'_W B\right) \rhd''_W B$$

  - “real comonads:”

$$\delta^*_{A,W,B} : A \rhd_W B \to \left(A \rhd'_W B\right) \rhd_W B$$

- derivation:

ideal comonads → comonads → “real comonads”

Monads and more

- three kinds of structures:
  - monads:

$$\eta_{A,W,B} : B \to A \rhd_W B$$
$$\mu_{A,W,B} : A \rhd_W (A \rhd_W B) \to A \rhd_W B$$

  - ideal monads:

$$\mu'_{A,W,B} : A \rhd'_W (A \rhd_W B) \to A \rhd'_W B$$

  - “gorgeous monads:”

$$\mu''_{A,W,B} : A \rhd''_W (A \rhd_W B) \to A \rhd''_W B$$

- derivation:

“gorgeous monads” → ideal monads → monads

Symmetric monoidal functor

- coherence map for the binary case:

$$A_1 \rhd''_{W_1} B_1 \times A_2 \rhd''_{W_2} B_2$$
$$\cong$$
$$(A_1 \times A_2) \rhd''_{W_1 \times W_2} ((A_1, W_1, B_1) \odot (A_2, W_2, B_2))$$

- definition of $\odot$:

$$(A_1, W_1, B_1) \odot (A_2, W_2, B_2)$$
$$=$$
$$\left(B_1 \times B_2\right) + \left(B_1 \times A_2 \rhd'_{W_2} B_2\right) + \left(A_1 \rhd'_{W_1} B_1 \times B_2\right)$$

- $W_1 \times W_2$ is minimum of W_1 and W_2
- coherence map for the nullary case:

$$1 \cong 1 \rhd''_1 0$$

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Conclusions and further work

- developed abstract process categories (APCs):
  - categorical models of an intuitionistic temporal logic with “until”
  - axiomatically defined
  - generalize temporal categories (Jeltsch 2012)
- further work:
  - recursion and corecursion on “until” proofs
  - integration with linear logic

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