

# Towards a Common Categorical Semantics for Linear-Time Temporal Logic and Functional Reactive Programming

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## Abstract

Linear-time temporal logic (LTL) and functional reactive programming (FRP) are related via a Curry–Howard correspondence. Based on this observation, we develop a common categorical semantics for a subset of LTL and its corresponding flavor of FRP. We devise a class of categorical models, called fan categories, that explicitly reflect the notion of time-dependent trueness of temporal propositions and a corresponding notion of time-dependent type inhabitation in FRP. Afterwards, we define the more abstract concept of temporal category by extending categorical models of intuitionistic S4. We show that fan categories are a special form of temporal categories.

*Keywords:* temporal logic, modal logic, functional reactive programming, categorical semantics

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## 1 Introduction

It was shown recently that there is a Curry–Howard correspondence between linear-time temporal logic (LTL) and functional reactive programming (FRP) [6,7,5]. This suggests that LTL and FRP can be given a common semantics. Category theory has been proven useful for modeling logics and programming calculi. So our goal is to define a class of categorical structures that can serve as models for LTL and for a corresponding FRP dialect. This paper describes our first results in this direction. We present the following contributions:

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- In [Section 2](#), we develop a class of categorical models for an intuitionistic temporal logic with a “globally” and a “finally” modality. We call these categorical models fan categories. Fan categories directly reflect the fact that trueness of temporal formulas depends on the time.
- In [Section 3](#), we demonstrate that fan categories are also models of FRP. We show that the time-dependent notion of trueness in temporal logic is related to time-dependent type inhabitation in FRP. We use our categorical semantics to explain the correspondence between the temporal modalities and the type constructors for behaviors and events, which are the key concepts of FRP.
- In [Section 4](#), we define the notion of intuitionistic S4 category based on earlier work by Kobayashi [8] and Bierman and de Paiva [3]. We prove that fan categories are a special form of S4 categories.
- In [Section 5](#), we introduce variants of the temporal modalities that refer only to the future. To reflect this in the semantics, we define ideal intuitionistic S4 categories. We prove a relationship between fan categories and ideal intuitionistic S4 categories that is analog to the result from [Section 4](#).
- In [Section 6](#), we extend ideal intuitionistic S4 categories with additional structure that captures the notion of linear time. We call the resulting structures temporal categories and prove that temporal categories cover fan categories as a special case.

We discuss related work in [Section 7](#) and give conclusions and an outlook on further work in [Section 8](#).

Throughout this paper, we will use certain notation when working with categorical products and coproducts. Let us define this notation, before starting with the payload of this paper.

**Definition 1.1 (Operations on products and coproducts)** For working with products and coproducts, we introduce notation as follows:

- Let  $I$  be an index set,  $\{A_i\}_{i \in I}$  be a family of objects for which a product exists, and  $\{f_i\}_{i \in I}$  be a family of morphisms  $f_i : B \rightarrow A_i$ . Then

$$\langle \{f_i\}_{i \in I} \rangle : B \rightarrow \prod_{i \in I} A_i$$

denotes the generalization of the binary product operation  $\langle \cdot, \cdot \rangle$  applied to the  $f_i$ . Furthermore for any  $i \in I$ ,  $\pi_i$  denotes the projection that corresponds to  $i$ .

- Expressions of the form  $[\{f_i\}_{i \in I}]$  and  $\iota_i$  for appropriate  $\{f_i\}$  and  $i$  denote the dual operations on coproducts.

- If we work in a bicartesian closed category (BCCC), then

$$\sigma : B \times \coprod_{i \in I} A_i \rightarrow \coprod_{i \in I} (B \times A_i)$$

denotes the natural transformation whose existence follows from the fact that BCCCs are distributive with respect to all coproducts. In the case of a binary coproduct, we add indices to  $\sigma$  that denote the objects involved, so that we have the following definition:

$$\begin{aligned} \sigma_{A,B,C} & : A \times (B + C) \rightarrow A \times B + A \times C \\ \sigma_{A,B,C} & := [\text{id}_A \times \iota_1, \text{id}_A \times \iota_2]^{-1} \end{aligned}$$

## 2 Temporal Logic and Fan Categories

We consider a temporal logic with a linear notion of time and with  $\Box$  (“globally”) and  $\Diamond$  (“finally”) as its only temporal operators. Since we want to have a Curry–Howard correspondence with FRP, our logic is intuitionistic instead of classical. Let  $P$  denote a set of atomic propositions. Then the syntax of formulas is given by the following BNF rule:

$$F ::= P \mid \top \mid \perp \mid F \wedge F \mid F \vee F \mid F \rightarrow F \mid \Box F \mid \Diamond F$$

In temporal logic, it depends on the time whether a formula is true or not. So intuitively, we can identify a formula  $\varphi$  of intuitionistic temporal logic with a function from times to formulas of intuitionistic propositional logic. We devise a class of categorical models that reflect this intuition. We call these models fan categories.

The standard categorical models of intuitionistic propositional logic are bicartesian closed categories (BCCCs). So we say that a fan category must be a product category  $\mathcal{C}^T$  where  $\mathcal{C}$  is a BCCC, and  $T$  is a set of times. An object of such a category is a function from  $T$  to  $\text{Obj } \mathcal{C}$ , and a morphism  $f : A \rightarrow B$  is a function that maps each time  $t$  to a morphism  $f(t) : A(t) \rightarrow B(t)$ . The latter means that for any temporal formulas  $\varphi$  and  $\psi$ , a proof of  $\varphi \vdash \psi$  shows that  $\varphi(t) \vdash \psi(t)$  holds for all times  $t$ .

The bicartesian closed structure of  $\mathcal{C}$  gives rise to a bicartesian closed structure of  $\mathcal{C}^T$ , where operations of the latter are just pointwise applications of the respective operations of  $\mathcal{C}$ . For example, the product-related operations of  $\mathcal{C}^T$  are defined as follows:

$$\begin{aligned} (A \times B)(t) & := A(t) \times B(t) & \pi_1(t) & := \pi_1 \\ \langle f, g \rangle(t) & := \langle f(t), g(t) \rangle & \pi_2(t) & := \pi_2 \end{aligned}$$

Clearly, the bicartesian closed structure of  $\mathcal{C}^T$  reflects the usual meanings of finite conjunctions, finite disjunctions, and implications in temporal logic.

For modeling the temporal modalities  $\Box$  and  $\Diamond$ , we equip our set  $T$  of times with a total order  $\leq$ , from which we derive orders  $<$ ,  $\geq$ , and  $>$  in the usual way. The intuition is that  $t < t'$  holds if at time  $t$ ,  $t'$  lies in the future. A formula  $\Box\varphi$  states that  $\varphi$  holds now and at every future time, while  $\Diamond\varphi$  states that  $\varphi$  holds now or at some future time. So a proposition  $(\Box\varphi)(t)$  corresponds to a (possibly infinite) conjunction of all  $\varphi(t')$  with  $t' \geq t$ , while a proposition  $(\Diamond\varphi)(t)$  corresponds to a disjunction of all such  $\varphi(t')$ .

Therefore, we model the modalities  $\Box$  and  $\Diamond$  by two functions  $\Box$  and  $\Diamond$  that turn objects into objects such that for any object  $A$ , the following holds:

$$(\Box A)(t) = \prod_{t' \geq t} A(t') \qquad (\Diamond A)(t) = \prod_{t' \geq t} A(t')$$

For this to work, we have to require that every family of objects of  $\mathcal{C}$  that is indexed by a set  $\{t' \mid t' \geq t\}$  admits a product and a coproduct. We actually demand a slightly stronger property, using index sets of the form  $\{t' \mid t' > t\}$ , because we will need this stronger property in [Section 5](#). We are now ready to give the definition of a fan category.

**Definition 2.1 (Fan category)** Let  $(T, \leq)$  be a totally ordered set, and let  $\mathcal{C}$  be a BCCC where every family of objects indexed by a set  $\{t' \mid t' > t\}$  has a product and a coproduct. The product category  $\mathcal{C}^T$  is then called a fan category.

The object mappings  $\Box$  and  $\Diamond$  can be turned into functors by defining the lifting of morphisms in the natural way.

**Definition 2.2 (Temporal functors of a fan category)** For each fan category  $\mathcal{C}^T$ , the temporal functors  $\Box$  and  $\Diamond$  are defined such that for every morphism  $f$  and every  $t \in T$ , the following equations hold:

$$(\Box f)(t) = \prod_{t' \geq t} f(t') \qquad (\Diamond f)(t) = \prod_{t' \geq t} f(t')$$

### 3 Connection to Functional Reactive Programming

The temporal logic we have defined in [Section 2](#) corresponds to a type system for FRP [\[6,7,5\]](#). Thereby, an FRP type corresponds to a temporal formula. Since such a formula can be seen as a function from times to formulas of intuitionistic propositional logic, an FRP type can be seen as a function from times to types of a simply typed  $\lambda$ -calculus with finite products and sums. So it depends on the time what values an FRP type inhabits.

Since fan categories are models of temporal logic, they are also models of FRP. If  $A$  is an object of a fan category that models an FRP type  $\tau$ , and  $t$  is a time,  $A(t)$  is the meaning of  $\tau(t)$ , that is, the simple type that corresponds to  $\tau$  at  $t$ . If  $A$  and  $B$  model FRP types  $\tau_1$  and  $\tau_2$ , a morphism from  $A$  to  $B$  models a family of functions from  $\tau_1$  to  $\tau_2$ , one for each time.

The key constructs of FRP are behaviors and events, which are used to describe temporal phenomena. A behavior is a time-varying value, while an event is an occurrence time with an attached value. The temporal modality  $\Box$  corresponds to a type constructor  $\Box$  for behaviors, while the modality  $\Diamond$  corresponds to a type constructor  $\Diamond$  for events. This can be seen by looking at the endofunctors  $\Box$  and  $\Diamond$ , which model the temporal modalities and hence also the type constructors that correspond to them. Remember that for any object  $A$ , we defined  $\Box A$  and  $\Diamond A$  as follows:

$$(\Box A)(t) := \prod_{t' \geq t} A(t') \qquad (\Diamond A)(t) := \coprod_{t' \geq t} A(t')$$

So an FRP type  $(\Box \tau)(t)$  corresponds to a (possibly infinite) product of all types  $\tau(t')$  with  $t' \geq t$ . This means that an inhabitant of  $(\Box \tau)(t)$  assigns a value of type  $\tau(t')$  to every time  $t' \geq t$  and thus characterizes a time-varying value of type  $\tau$ . Likewise, a type  $(\Diamond \tau)(t)$  corresponds to a sum of all types  $\tau(t')$  with  $t' \geq t$ . So an inhabitant of  $(\Diamond \tau)(t)$  is a pair of a time  $t' \geq t$  and a value of type  $\tau(t')$  and thus characterizes an occurrence time with an attached value of type  $\tau$ .

## 4 Connection to Models of Intuitionistic S4

The classical modal logic S4 corresponds to the class of Kripke frames whose accessibility relation is a preorder. Classical temporal logics with a linear notion of time use totally ordered sets of times as Kripke frames. So a Kripke model for such a logic is also a Kripke model for S4. It is reasonable to assume that a similar connection exists in the case of intuitionistic logics and categorical models. In this section, we show that this is in fact the case.

Categorical models for intuitionistic S4 variants are studied by Kobayashi [8] as well as by Bierman and de Paiva [3]. We define the notion of intuitionistic S4 category based on their work and show that fan categories give rise to intuitionistic S4 categories.

**Definition 4.1 (Cartesian comonad)** Let  $\mathcal{C}$  be a category with finite products. A tuple  $(U, \varepsilon, \delta, m, n)$  is a cartesian comonad on  $\mathcal{C}$  if  $(U, \varepsilon, \delta)$  is a comonad on  $\mathcal{C}$ , and  $(U, m, n)$  is a cartesian endofunctor on  $\mathcal{C}$ , that is, a strong monoidal functor from the monoidal category  $(\mathcal{C}, \times, 1)$  to itself.

$$\begin{array}{ccccc}
 & \xrightarrow{\lambda_{TA}^{-1}} & & \xrightarrow{\alpha_{UA,UB,TC}^{-1}} & \\
 TA & & 1 \times TA & & UA \times (UB \times TC) & & (UA \times UB) \times TC \\
 \downarrow T\lambda_A^{-1} & & \downarrow n \times \text{id}_{TA} & & \downarrow \text{id}_{UA} \times s_{B,C} & & \downarrow m_{A,B} \times \text{id}_{TC} \\
 & & U1 \times TA & & UA \times T(UB \times C) & & U(A \times B) \times TC \\
 & & \downarrow s_{1,A} & & \downarrow s_{A,UB \times C} & & \downarrow s_{A \times B, C} \\
 T(1 \times A) & & T(U1 \times A) & & T(UA \times (UB \times C)) & & T(U(A \times B) \times C) \\
 & \xrightarrow{T(n \times \text{id}_A)} & & & \xrightarrow{T\alpha_{UA,UB,C}^{-1}} & & \xrightarrow{T(m_{A,B} \times \text{id}_C)} \\
 & & & & T((UA \times UB) \times C) & & T(U(A \times B) \times C)
 \end{array}$$

Fig. 1. Compatibility of tensorial strength with a cartesian endofunctor

$$\begin{array}{ccccc}
 & \xrightarrow{\delta_A \times \text{id}_{TB}} & & \xrightarrow{\text{id}_{UA} \times \eta_B} & \xrightarrow{\text{id}_{UA} \times \mu_B} & \\
 UA \times TB & & UUA \times TB & & UA \times B & & UA \times TB & & UA \times TTB \\
 \downarrow s_{A,B} & & \downarrow s_{UA,B} & & \downarrow \eta_{UA \times B} & & \downarrow s_{A,B} & & \downarrow s_{A,TB} \\
 T(UA \times B) & & T(UUA \times B) & & T(UA \times B) & & T(UA \times TB) & & T(UA \times TTB) \\
 & \xrightarrow{T(\delta_A \times \text{id}_B)} & & & & & \downarrow Ts_{A,B} & & \\
 & & & & & & TT(UA \times B) & & \\
 & & & & & & \xrightarrow{\mu_{UA \times B}} & & 
 \end{array}$$

Fig. 2. Compatibility of tensorial strength with a comonad and a monad

**Definition 4.2 ( $\mathcal{U}$ -strong monad)** Let  $\mathcal{C}$  be a category with finite products and  $\mathcal{U} = (U, \varepsilon, \delta, m, n)$  be a cartesian comonad on  $\mathcal{C}$ . A tuple  $(T, \eta, \mu, s)$  is a  $\mathcal{U}$ -strong monad if  $(T, \eta, \mu)$  is a monad on  $\mathcal{C}$ ,  $s$  is a natural transformation with  $s_{A,B} : UA \times TB \rightarrow T(UA \times B)$ , and the diagrams in Figures 1 and 2 commute. The transformation  $s$  is called tensorial strength.

**Definition 4.3 (Intuitionistic S4 category)** An intuitionistic S4 category is a tuple  $(\mathcal{C}, \square, \varepsilon, \delta, m, n, \diamond, \eta, \mu, s)$  where  $\mathcal{C}$  is a BCCC,  $\mathcal{U} = (\square, \varepsilon, \delta, m, n)$  is a cartesian comonad on  $\mathcal{C}$ , and  $(\diamond, \eta, \mu, s)$  is a  $\mathcal{U}$ -strong monad.

Kobayashi [8] defines the notion of CS4 structure, which is very similar to our notion of intuitionistic S4 category. The difference is that a CS4 structure may only have weak coproducts instead of proper coproducts, and that the

$$\begin{aligned}
 \varepsilon(t) &: \prod_{t' \geq t} A(t') \rightarrow A(t) & \delta(t) &: \prod_{t' \geq t} A(t') \rightarrow \prod_{t' \geq t} \prod_{t'' \geq t'} A(t'') \\
 \varepsilon(t) &:= \pi_t & \delta(t) &:= \langle \{ \{ \{ \pi_{t''} \}_{t'' \geq t'} \} \}_{t' \geq t} \rangle
 \end{aligned}$$

Fig. 3. Comonad structure of a fan category

$$\begin{aligned}
 m(t) &: \prod_{t' \geq t} A(t') \times \prod_{t' \geq t} B(t') \rightarrow \prod_{t' \geq t} (A(t') \times B(t')) & n(t) &: 1 \rightarrow \prod_{t' \geq t} 1 \\
 m(t) &:= \langle \{ \pi_{t'} \times \pi_{t'} \}_{t' \geq t} \rangle & n(t) &:= \langle \{ \text{id}_1 \}_{t' \geq t} \rangle
 \end{aligned}$$

Fig. 4. Cartesian endofunctor structure of a fan category

$$\begin{aligned}
 \eta(t) &: A(t) \rightarrow \prod_{t' \geq t} A(t') & \mu(t) &: \prod_{t' \geq t} \prod_{t'' \geq t'} A(t'') \rightarrow \prod_{t' \geq t} A(t') \\
 \eta(t) &:= \iota_t & \mu(t) &:= [\{ \{ \{ \iota_{t''} \}_{t'' \geq t'} \} \}_{t' \geq t}]
 \end{aligned}$$

Fig. 5. Monad structure of a fan category

functor  $\diamond$  must preserve weak initial objects. Kobayashi probably needs this, because his logic has  $\diamond \perp \rightarrow \perp$  as a theorem. In FRP terms, this would mean that there is a function of type  $\diamond 0 \rightarrow 0$ , that is, a function that can yield a non-existing value of type 0 now, although such a value is only promised to be available at some time that may not have been reached yet. Clearly, such a function cannot exist. Since we want to maintain a Curry–Howard correspondence between temporal logic and FRP, we reject  $\diamond \perp \rightarrow \perp$ .

Bierman and de Paiva [3] define categorical models for the intuitionistic modal logic IS4. In contrast to us, they do not require the monoidal endofunctor  $(\square, m, n)$  to be strong. However, they enforce certain coherence conditions between the monoidal functor structure and the comonad structure, which hold automatically for a strong monoidal functor. In Section 10 of their paper, they discuss some possible extra conditions related to the maps  $!_{\square A} : \square A \rightarrow 1$  and  $\Delta_{\square A} : \square A \rightarrow \square A \times \square A$ . These conditions also hold automatically if  $(\square, m, n)$  is strong, as is the case in our intuitionistic S4 categories. Furthermore, their coherence conditions for tensorial strength differ from what we have depicted in Figures 1 and 2.

We will now state and prove the relationship between fan categories and intuitionistic S4 categories that we mentioned at the beginning of [this section](#).

**Theorem 4.4** *If  $\mathcal{C}^T$  is a fan category, and  $\square$  and  $\diamond$  are its temporal functors, then there are natural transformations  $\varepsilon$ ,  $\delta$ ,  $m$ ,  $n$ ,  $\eta$ ,  $\mu$ , and  $s$  such that  $(\mathcal{C}, \square, \varepsilon, \delta, m, n, \diamond, \eta, \mu, s)$  is an intuitionistic S4 category.*

**Proof.** We construct the abovementioned natural transformations as shown in Figures 3 through 6. Proving that these transformations fulfill the necessary conditions is straightforward and therefore left out here.  $\square$

$$\begin{aligned}
 s(t) &: \prod_{t' \geq t} A(t') \times \prod_{t' \geq t} B(t') \rightarrow \prod_{t' \geq t} \left( \prod_{t'' \geq t'} A(t'') \times B(t') \right) \\
 s(t) &:= \left( \prod_{t' \geq t} (\langle \{\pi_{t''}\}_{t'' \geq t'} \rangle \times \text{id}_{B(t')}) \right) \sigma
 \end{aligned}$$

Fig. 6. Tensorial strength of a fan category

## 5 Future Only

A proposition  $\Box\varphi$  of our temporal logic forces  $\varphi$  to hold also at the current time. Likewise, a proposition  $\Diamond\varphi$  allows  $\varphi$  to hold at the current time instead of in the future. However, there are cases where modalities that only refer to the future are desired. In LTL, where we have a discrete notion of time and a “next” modality  $\bigcirc$ , we can define future-only variants of  $\Box$  and  $\Diamond$  as follows:

$$\Box'\varphi := \bigcirc\Box\varphi \qquad \Diamond'\varphi := \bigcirc\Diamond\varphi$$

In our logic, where time is not necessarily discrete, it is not possible to derive  $\Box'$  and  $\Diamond'$  from  $\Box$  and  $\Diamond$ . So it is worthwhile to introduce  $\Box'$  and  $\Diamond'$  as the fundamental modalities and define  $\Box$  and  $\Diamond$  in terms of them as follows:

$$\Box\varphi := \varphi \wedge \Box'\varphi \qquad \Diamond\varphi := \varphi \vee \Diamond'\varphi$$

This increases the expressiveness of our logic. Expressiveness of FRP can be increased in an analog way. In [the next section](#), we will use the additional expressiveness that  $\Diamond'$  gives us.

We define a variant of intuitionistic S4 categories that also models the future-only modalities. We introduce two new endofunctors  $\Box'$  and  $\Diamond'$ , and derive  $\Box$  and  $\Diamond$  from them as follows:

$$\Box A := A \times \Box' A \qquad \Diamond A := A + \Diamond' A$$

According to [Definition 4.3](#), we need a comonad structure for  $\Box$  and a monad structure for  $\Diamond$ . The natural way to get these is to add an ideal comonad structure for  $\Box'$  and an ideal monad structure for  $\Diamond'$ .

**Definition 5.1 (Ideal comonad)** A pair  $(U', \delta')$  is an ideal comonad on a category  $\mathcal{C}$  with binary products if  $U'$  is an endofunctor on  $\mathcal{C}$ ,  $\delta'$  is a natural transformation from  $U'$  to  $U'(\text{Id} \times U')$ , and  $(\text{Id} \times U', \pi_1, \langle \text{id}, \delta' \pi_2 \rangle)$  is a comonad.

**Definition 5.2 (Ideal monad)** A pair  $(T', \mu')$  is an ideal monad on a category  $\mathcal{C}$  with binary coproducts if  $T'$  is an endofunctor on  $\mathcal{C}$ ,  $\mu'$  is a natural transformation from  $T'(\text{Id} + T')$  to  $T'$ , and  $(\text{Id} + T', \iota_1, [\text{id}, \iota_2 \mu'])$  is a monad.

From an ideal comonad  $(\Box', \delta')$  and an ideal monad  $(\Diamond', \mu')$ , we can derive the comonad  $(\Box, \varepsilon, \delta)$  and the monad  $(\Diamond, \eta, \mu)$  that we need for an intuitionistic



S4 category. We also want to derive the natural transformations  $m$ ,  $n$ , and  $s$  from more basic transformations that work with  $\square'$  and  $\diamond'$ . For this, we introduce the two new concepts of ideal cartesian comonad ([Definition 5.3](#)) and  $\mathcal{U}'$ -strong ideal monad ([Definition 5.5](#)).

**Definition 5.3 (Ideal cartesian comonad)** Let  $\mathcal{C}$  be a category with finite products. A tuple  $(U', \delta', m', n')$  is an ideal cartesian comonad on  $\mathcal{C}$  if  $(U', \delta')$  is an ideal comonad on  $\mathcal{C}$ , and  $(U', m', n')$  is a cartesian endofunctor on  $\mathcal{C}$ .

**Lemma 5.4** *If  $(U', \delta', m', n')$  is an ideal cartesian comonad on a category  $\mathcal{C}$  with finite products, then*

$$(\text{Id} \times U', \pi_1, \langle \text{id}, \delta' \pi_2 \rangle, \langle \pi_1 \times \pi_1, m'(\pi_2 \times \pi_2) \rangle, \langle \text{id}_1, n' \rangle)$$

*is a cartesian comonad on  $\mathcal{C}$ .*

**Proof.**  $(\text{Id} \times U', \pi_1, \langle \text{id}, \delta' \pi_2 \rangle)$  is a comonad on  $\mathcal{C}$  according to [Definition 5.1](#). Checking that  $(\text{Id} \times U', \langle \pi_1 \times \pi_1, m'(\pi_2 \times \pi_2) \rangle, \langle \text{id}_1, n' \rangle)$  is a cartesian endofunctor is straightforward.  $\square$

**Definition 5.5 ( $\mathcal{U}'$ -strong ideal monad)** Let  $\mathcal{C}$  be a distributive category,  $\mathcal{U}' = (U', \delta', m', n')$  be an ideal cartesian comonad on  $\mathcal{C}$  and  $\mathcal{U} = (U, \varepsilon, \delta, m, n)$  be the cartesian comonad induced by  $\mathcal{U}'$  according to [Lemma 5.4](#). A tuple  $(T', \mu', s')$  is a  $\mathcal{U}'$ -strong ideal monad if the following conditions hold:

- $(T', \mu')$  is an ideal monad on  $\mathcal{C}$ .
- $s'$  is a natural transformation with  $s'_{A,B} : U'A \times T'B \rightarrow T'(UA \times B)$ .
- If  $(T, \eta, \mu)$  is the monad induced by  $(T', \mu')$  according to [Definition 5.2](#), and the natural transformation  $s$  is defined by

$$\begin{aligned} s_{A,B} &: UA \times TB \rightarrow T(UA \times B) \\ s_{A,B} &:= (\text{id}_{UA \times B} + s'_{A,B}(\pi_2 \times \text{id}_{T'B}))\sigma_{UA,B,T'B} \end{aligned}$$

then  $(T, \eta, \mu, s)$  is a  $\mathcal{U}$ -strong monad.

**Definition 5.6 (Ideal intuitionistic S4 category)** An ideal intuitionistic S4 category is a tuple  $(\mathcal{C}, \square', \delta', m', n', \diamond', \mu', s')$  where  $\mathcal{C}$  is a BCCC,  $\mathcal{U}' = (\square', \delta', m', n')$  is an ideal cartesian comonad on  $\mathcal{C}$ , and  $(\diamond', \mu', s')$  is a  $\mathcal{U}'$ -strong ideal monad.

From [Lemma 5.4](#) and [Definition 5.5](#), it is immediately clear that every ideal intuitionistic S4 category gives rise to an intuitionistic S4 category. Another important property is that fan categories give rise to ideal intuitionistic S4 categories. We define the ideal temporal functors of a fan category analogously to [Definition 2.2](#) and obtain a fact similar to the one of [Theorem 4.4](#).

**Definition 5.7 (Ideal Temporal Functors of a Fan Category)** Let  $\mathcal{C}^T$  be a fan category. The ideal temporal functors  $\square'$  and  $\diamond'$  of  $\mathcal{C}^T$  are defined such that for every morphism  $f$  and every  $t \in T$ , the following equations hold:

$$(\square' f)(t) = \coprod_{t' > t} f(t') \qquad (\diamond' f)(t) = \coprod_{t' > t} f(t')$$

**Theorem 5.8** *If  $\mathcal{C}^T$  is a fan category, and  $\square'$  and  $\diamond'$  are its ideal temporal functors, then there are natural transformations  $\delta'$ ,  $m'$ ,  $n'$ ,  $\mu'$ , and  $s'$  such that  $(\mathcal{C}, \square', \delta', m', n', \diamond', \mu', s')$  is an ideal intuitionistic S4 category.*

**Proof.** We define the natural transformations  $\delta'$ ,  $m'$ ,  $n'$ ,  $\mu'$ , and  $s'$  by taking the definitions of  $\delta$ ,  $m$ ,  $n$ ,  $\mu$ , and  $s$  from the proof of [Theorem 4.4](#) and replacing  $\geq$  by  $>$  wherever we now deal with  $\square'$  and  $\diamond'$  instead of  $\square$  and  $\diamond$ . In the following, we show that these definitions lead in fact to an ideal intuitionistic S4 category.

We derive functors  $\square$  and  $\diamond$  and natural transformations  $\varepsilon$ ,  $\delta$ ,  $m$ ,  $n$ ,  $\eta$ ,  $\mu$ , and  $s$  from  $\square'$ ,  $\diamond'$ ,  $\delta'$ ,  $m'$ ,  $n'$ ,  $\mu'$ , and  $s'$  according to [Lemma 5.4](#) and [Definitions 5.2](#) and [5.5](#). The functors  $\square$  and  $\diamond$  are isomorphic to the temporal functors from [Definition 2.2](#), and the natural transformations are the ones defined in the proof of [Theorem 4.4](#) up to isomorphism. So they form an intuitionistic S4 category. This means that  $\mathcal{U} = (\square, \varepsilon, \delta, m, n)$  is a cartesian comonad, and  $(\diamond, \eta, \mu, s)$  is a  $\mathcal{U}$ -strong monad. As a result,  $\mathcal{U}' = (\square', \delta', m', n')$  is an ideal cartesian comonad, and  $(\diamond', \mu', s')$  is a  $\mathcal{U}'$ -strong ideal monad. This proves the claim.  $\square$

## 6 Linear Time

Fan categories are rather concrete. In this section, we develop a much more abstract notion of categorical model for temporal logic and FRP. Ideal intuitionistic S4 categories are a good starting point for this undertaking. Their problem is that they do not capture the notion of linear time. This is analog to Kripke models of classical logics, where S4 permits arbitrary preorders as Kripke frames, while linear-time temporal logics only permit total orders. In this section, we enrich ideal intuitionistic S4 categories with further structure that reflects the linearity of time. We call the resulting constructs temporal categories.

In order to see how we can encode linearity of time in our categorical models, let us first look at FRP. Linearity of time is ensured if there is a function **race** of type

$$\diamond\tau_1 \times \diamond\tau_2 \rightarrow \diamond(\tau_1 \times \tau_2 + \tau_1 \times \diamond'\tau_2 + \diamond'\tau_1 \times \tau_2) \quad .$$

In the following, we will explain why this is the case.

Let  $e_1$  and  $e_2$  be events of types  $\diamond\tau_1$  and  $\diamond\tau_2$ , and let  $t_1$ ,  $t_2$ , and  $t$  be the times at which  $e_1$ ,  $e_2$ , and  $\mathbf{race}(e_1, e_2)$  fire. If  $\mathbf{race}(e_1, e_2)$  contains a value of type  $\tau_1 \times \tau_2$ , the components of this pair must come from  $e_1$  and  $e_2$ , because the values that  $e_1$  and  $e_2$  carry are the only values of types  $\tau_1$  and  $\tau_2$  that are available to  $\mathbf{race}$ . Since our FRP dialect generally does not allow us to shift values to different times, we have  $t = t_1 = t_2$ . If  $\mathbf{race}(e_1, e_2)$  contains a value of type  $\tau_1 \times \diamond'\tau_2$  or  $\diamond'\tau_1 \times \tau_2$ , we get a remainder event of a type  $\diamond'\tau_i$ , which fires after  $t$ . Since it contains a value of type  $\tau_i$ , it fires at  $t_i$ . So the second and the third alternative correspond to the conditions  $t = t_1 < t_2$  and  $t = t_2 < t_1$ , respectively. All in all, we now know that one of the three alternatives  $t_1 = t_2$ ,  $t_1 < t_2$ , and  $t_1 > t_2$  holds, which ensures that time is linear. We furthermore know that  $\mathbf{race}(e_1, e_2)$  fires at time  $\min(t_1, t_2)$ .

Let us now turn to category theory again. Say we have an ideal intuitionistic S4 category  $(\mathcal{C}, \square', \delta', m', n', \diamond', \mu', s')$ , which induces an intuitionistic S4 category  $(\mathcal{C}, \square, \varepsilon, \delta, m, n, \diamond, \eta, \mu, s)$ . We define a binary operation  $\odot$  on objects as follows:

$$A \odot B := A \times B + A \times \diamond' B + \diamond' A \times B$$

To give a meaning to  $\mathbf{race}$ , we require that for any morphisms  $f : C \rightarrow \diamond A$  and  $g : C \rightarrow \diamond B$ , there is a morphism  $\langle\langle f, g \rangle\rangle : C \rightarrow \diamond(A \odot B)$ . We realize this by requiring that for any objects  $A$  and  $B$ ,  $A \odot B$  is a product of  $A$  and  $B$  in the Kleisli category of the monad  $(\diamond, \eta, \mu)$ .

For a proper product structure, we also need projections, which we call  $\varpi_1$  and  $\varpi_2$  in order to not confuse them with the projections  $\pi_1$  and  $\pi_2$  of the original category  $\mathcal{C}$ . The projections  $\varpi_i$  have the types  $C_1 \odot C_2 \rightarrow C_i$  in the Kleisli category. So they have the types  $C_1 \odot C_2 \rightarrow \diamond C_i$  in the original category, which are the same as

$$C_1 \times C_2 + C_1 \times \diamond' C_2 + \diamond' C_1 \times C_2 \rightarrow C_i + \diamond' C_i \quad .$$

The straightforward definition of the  $\varpi_i$  is  $\varpi_i := [\iota_1 \pi_i, \iota_i \pi_i, \iota_{1-i} \pi_i]$ .

For  $\odot$ ,  $\langle\langle \cdot, \cdot \rangle\rangle$ ,  $\varpi_1$ , and  $\varpi_2$  to form a product, the following equations must hold in the Kleisli category for all suitable  $h_1$ ,  $h_2$ , and  $h$ :

$$\varpi_i \langle\langle h_1, h_2 \rangle\rangle = h_i \qquad \langle\langle \varpi_1 h, \varpi_2 h \rangle\rangle = h$$

This means that the following equations must hold in the original category  $\mathcal{C}$ :

$$\mu(\diamond \varpi_i) \langle\langle h_1, h_2 \rangle\rangle = h_i \qquad \langle\langle \mu(\diamond \varpi_1) h, \mu(\diamond \varpi_2) h \rangle\rangle = h$$

Looking at FRP, the first equation tells us that we can recover the events  $e_i$  from a value  $\mathbf{race}(e_1, e_2)$  using functions  $\mathbf{recover}_i$  that correspond to the transformations  $\mu(\diamond \varpi_i)$ . The second equation states that every value  $e$  from the codomain of  $\mathbf{race}$  can be constructed by applying  $\mathbf{race}$  to  $(\mathbf{recover}_1 e, \mathbf{recover}_2 e)$ .

We now define temporal categories by extending ideal intuitionistic S4 categories as described above.

**Definition 6.1 (Temporal category)** Say  $\mathcal{M} = (\mathcal{C}, \square', \delta', m', n', \diamond', \mu', s')$  is an ideal intuitionistic S4 category, and  $(\mathcal{C}, \square, \varepsilon, \delta, m, n, \diamond, \eta, \mu, s)$  is the intuitionistic S4 category induced by it. For all objects  $A$  and  $B$ , let  $A \odot B$  be defined by

$$A \odot B := A \times B + A \times \diamond' B + \diamond' A \times B ,$$

and for all objects  $C_1$  and  $C_2$  and all  $i \in \{1, 2\}$ , let  $\varpi_i$  be defined as follows:

$$\begin{aligned} \varpi_i &: C_1 \odot C_2 \rightarrow \diamond C_i \\ \varpi_i &:= [\iota_1 \pi_i, \iota_i \pi_i, \iota_{1-i} \pi_i] \end{aligned}$$

$\mathcal{M}$  is a temporal category if each  $A \odot B$  is a product of  $A$  and  $B$  with projections  $\varpi_1$  and  $\varpi_2$  in the Kleisli category of  $(\diamond, \eta, \mu)$ .

[Definition 4.2](#) enforces certain relationships between a strength transformation  $s$  and other natural transformations, which are depicted in [Figures 1](#) and [2](#). The reader might wonder why we did not specify similar relationships between  $\langle\langle \cdot, \cdot \rangle\rangle$  and the transformations  $\mu'$  and  $s'$  in [Definition 6.1](#). We did not do so, since we strongly conjecture that any such coherence conditions that are sensible already follow from [Definition 6.1](#) as it is. Making this claim precise and proving it is a possible goal for the future.

Our final result is that temporal categories are indeed a generalization of fan categories.

**Theorem 6.2** *If  $\mathcal{C}^T$  is a fan category, and  $\square'$  and  $\diamond'$  are its ideal temporal functors, then there are natural transformations  $\delta', m', n', \mu',$  and  $s'$  such that  $(\mathcal{C}, \square', \delta', m', n', \diamond', \mu', s')$  is a temporal category.*

**Proof.** We construct the abovementioned natural transformations like we did in the proof of [Theorem 5.8](#). So we know that  $(\mathcal{C}, \square', \delta', m', n', \diamond', \mu', s')$  is an ideal intuitionistic S4 category.

We now define the operation  $\langle\langle \cdot, \cdot \rangle\rangle$ . We first introduce a helper morphism  $\theta_t$  for every time  $t$ :

$$\begin{aligned} \theta_t &: \prod_{t_1 \geq t} A(t_1) \times \prod_{t_2 \geq t} B(t_2) \rightarrow \prod_{t_1 \geq t} \prod_{t_2 \geq t} (A(t_1) \times B(t_2)) \\ \theta_t &:= \left( \prod_{t_1 \geq t} \sigma \langle \pi_2, \pi_1 \rangle \right) \sigma \langle \pi_2, \pi_1 \rangle \end{aligned}$$

We furthermore define transformations  $\kappa_{t,t_1,t_2}$  for times  $t, t_1,$  and  $t_2$  with  $t \leq t_1$

and  $t \leq t_2$ :

$$\begin{aligned} \kappa_{t,t_1,t_2} &: A(t_1) \times B(t_2) \rightarrow \prod_{t' \geq t} (A \odot B)(t') \\ \kappa_{t,t_1,t_2} &:= \begin{cases} \iota_{t_1} \iota_1 & \text{if } t_1 = t_2 \\ \iota_{t_1} \iota_2 (\text{id}_{A(t_1)} \times \iota_{t_2}) & \text{if } t_1 < t_2 \\ \iota_{t_2} \iota_3 (\iota_{t_1} \times \text{id}_{B(t_2)}) & \text{if } t_1 > t_2 \end{cases} \end{aligned}$$

Finally, we define  $\langle\langle f, g \rangle\rangle$  for any morphisms  $f : C \rightarrow \diamond A$  and  $g : C \rightarrow \diamond B$  as follows:

$$\begin{aligned} \langle\langle f, g \rangle\rangle(t) &: C(t) \rightarrow \prod_{t' \geq t} (A \odot B)(t') \\ \langle\langle f, g \rangle\rangle(t) &:= [\{\{\{\kappa_{t,t_1,t_2}\}_{t_2 \geq t}\}_{t_1 \geq t}\}] \theta_t \langle f(t), g(t) \rangle \end{aligned}$$

We leave it as an exercise to the reader to show that in the Kleisli category,  $\varpi_i \langle\langle h_1, h_2 \rangle\rangle = h_i$  and  $\langle\langle \varpi_1 h, \varpi_2 h \rangle\rangle = h$  hold for all suitable  $h_1$ ,  $h_2$ , and  $h$ . This then completes the proof.  $\square$

## 7 Related Work

Jeffrey [5] presents an implementation of FRP in the dependently typed programming language Agda. Based on this, he develops a category **RSet**, which expresses the notion of time-dependent type inhabitation and is thus strongly related to our fan categories. Jeffrey also uses FRP analogs of advanced temporal operators to develop variants of **RSet** that enforce causality of FRP operations.

In Section 4, we discussed the relationships between our intuitionistic S4 categories and the categorical models by Kobayashi [8] and by Bierman and de Paiva [3]. Alechina et. al. [1] show how the latter are related to algebraic models and Kripke models.<sup>3</sup>

Bellin et. al. [2] study an intuitionistic version IK of the basic modal logic K. They also define categorical models of IK. These models lack the comonadic and monadic structure that intuitionistic S4 categories possess, and use a tensorial strength transformation  $t$  with

$$t_{A,B} : \square A \times \diamond B \rightarrow \diamond(A \times B)$$

<sup>3</sup> Note that there is a slight confusion in terminology. Kobayashi calls his S4 variant CS4, Bierman and de Paiva call theirs IS4, but Alechina et. al. use the name CS4 for the logic of Bierman and de Paiva.

instead of the strength transformation  $s$  with

$$s_{A,B} : \Box A \times \Diamond B \rightarrow \Diamond(\Box A \times B) \quad .$$

In intuitionistic S4 categories, we can derive  $t$  from  $s$  by  $t := (\Diamond(\varepsilon_A \times \text{id}_B))s$ . As a result, every model of IK is also an intuitionistic S4 category. The lack of structure in IK models corresponds to a lack of axioms in logic. Classically, this lack of axioms corresponds to the fact that K corresponds to the class of all Kripke frames, while S4 corresponds to the class of Kripke frames where the accessibility relation is a preorder.<sup>4</sup>

Krishnaswami and Benton [9] give an FRP semantics based on the category of 1-bounded ultrametric spaces, and Birkedal et. al [4] study the related category of presheaves over the natural numbers. Both approaches use a discrete notion of time, while our work is compatible with any totally ordered set of times. Studying the connections between our developments and the ones of Krishnaswami and Benton as well as the ones of Birkedal et. al. remains a task for the future.

## 8 Conclusions and Further Work

We have defined fan categories, which are categorical models of a subset of an intuitionistic LTL variant and a corresponding flavor of FRP. Fan categories directly express the notion of time-dependent trueness in LTL and the related notion of time-dependent type inhabitation of our FRP dialect. We have furthermore defined the more abstract notion of temporal category based on categorical models of intuitionistic S4 and shown that fan categories are a specialization of temporal categories.

In a future publication, we want to extend temporal categories such that they also cover other modalities of LTL and their FRP counterparts. Furthermore, we want to study how recursion can be integrated into categorical models of FRP. We also want to use concepts from temporal categories in the interface design and possibly the implementation of FRP systems. Furthermore, we are interested in combining temporal logic with other kinds of logic and studying the corresponding programming paradigms. Another task is to find out about relationships to other categorical FRP semantics, as discussed in [Section 7](#).

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<sup>4</sup> Note that the axiom  $\Box\varphi \rightarrow \varphi$  ensures reflexivity, and the axiom  $\Box\varphi \rightarrow \Box\Box\varphi$  ensures transitivity. Analogously,  $\varphi \rightarrow \Diamond\varphi$  ensures reflexivity, and  $\Diamond\Diamond\varphi \rightarrow \Diamond\varphi$  ensures transitivity.

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