Towards a Common Categorical Semantics for Linear-Time Temporal Logic and Functional Reactive Programming

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Temporal logic

- intuitionistic temporal logic with temporal operators □ and ◇:

\[ F ::= A \mid \top \mid \bot \mid F \land F \mid F \lor F \mid F \rightarrow F \mid \Box F \mid \Diamond F \]

- dependance on time:
  - time-dependent whether a proposition is true
  - time-dependent whether a certain proof proves a proposition

- meaning of □ and ◇:
  - □ϕ ϕ holds at current and every future time
  - ◇ϕ ϕ holds at current or some future time
Functional reactive programming

- functional programming with additional type constructors
  - □ and ◊:
    \[
    T ::= A \mid 1 \mid 0 \mid T \times T \mid T + T \mid T \to T \mid □T \mid ◊T
    \]

- dependance on time:
  - time-dependent whether a type is inhabited
  - time-dependent whether a certain value inhabits a type

- inhabitants of □ and ◊:
  - □τ time-varying value of type τ (behavior)
  - ◊τ time and associated value of type τ (event)
Simple categorical semantics

- ingredients of a categorical model:
  \((T, \leq)\) totally ordered set of times
  \(C\) bicartesian closed category (BCCC)

- \(C^T\) is a categorical model of temporal logic:
  - object \(A\) maps times \(t\) to objects \(A(t)\) of \(C\)
  - \(f : A \rightarrow B\) maps times \(t\) to morphisms \(f(t) : A(t) \rightarrow B(t)\)

- endofunctors \(\Box\) and \(\Diamond\) defined as follows:
  \[
  (\Box A)(t) := \prod_{t' \geq t} A(t') \\
  (\Diamond A)(t) := \bigsqcup_{t' \geq t} A(t')
  \]

- possibly some infinite products and coproducts must exist in \(C\)

Goal

axiomatic semantics that covers this semantics as a special case
Inspiration

Satoshi Kobayashi
*Monad as Modality*
Theoretical Computer Science 175 (1997), pp. 29–74

Gavin Bierman and Valeria de Paiva
*On an Intuitionistic Modal Logic*
Basic structure

- bicartesian closed categories as the basis
- intuition of time independance:
  - $f : \llbracket \varphi \rrbracket \to \llbracket \psi \rrbracket$ models a proof showing that $\varphi$ implies $\psi$ at every time
  - $f : \llbracket \tau_1 \rrbracket \to \llbracket \tau_2 \rrbracket$ models a function from $\tau_1$ to $\tau_2$ that works at every time
- addition of endofunctors $\Box$ and $\Diamond$
- gives us functor applications:

\[
\begin{array}{c}
\frac{f : A \to B}{\Box f : \Box A \to \Box B} \\
\frac{f : A \to B}{\Diamond f : \Diamond A \to \Diamond B}
\end{array}
\]
Monoidal functors

- □ is a strong monoidal functor on the cartesian structure (cartesian functor):
  \[ □A \times □B ≅ □(A \times B) \]
  \[ 1 ≅ □1 \]

- ◊ is not a strong monoidal functor on the cocartesian structure:
  - natural transformations of these types would have to exist:
    \[ ◊(A + B) \to ◊A + ◊B \]
    \[ ◊0 \to 0 \]
  - correspond to non-causal functions in FRP
Comonads, monads, and tensorial strength

- $\Box$ is a comonad:
  \[ \varepsilon_A : \Box A \to A \]
  \[ \delta_A : \Box A \to \Box \Box A \]

- $\Diamond$ is a monad:
  \[ \eta_A : A \to \Diamond A \]
  \[ \mu_A : \Diamond \Diamond A \to \Diamond A \]

- $\Diamond$ is $\Box$-strong:
  \[ s_{A,B} : \Box A \times \Diamond B \to \Diamond (\Box A \times B) \]
functors $\Box'$ and $\Diamond'$ with the following properties:

\[
\Box A = A \times \Box' A
\]
\[
\Diamond A = A + \Diamond' A
\]

- $\Box'$ is an ideal comonad:

\[
\delta'_A : \Box' A \to \Box' \Box A
\]

- $\Diamond'$ is an ideal monad:

\[
\mu'_A : \Diamond' \Diamond A \to \Diamond' A
\]
Linear time

- require existence of a natural transformation $r$ with
  \[ r_{A,B} : \Diamond A \times \Diamond B \rightarrow \Diamond (A \circ B) \]

- definition of $\circ$:
  \[ A \circ B := A \times B + A \times \Diamond' B + \Diamond' A \times B \]

- alternatives of $A \circ B$ reflect relations between
  the time $t_A$ of $\Diamond A$ and the time $t_B$ of $\Diamond B$:
  \[ A \times B \quad t_A = t_B \]
  \[ A \times \Diamond' B \quad t_A < t_B \]
  \[ \Diamond' A \times B \quad t_A > t_B \]

- linearity of time is guaranteed:
  \[ (t_A = t_B) \lor (t_A < t_B) \lor (t_A > t_B) \]

- time of $\Diamond (A \circ B)$ is the minimum of the above times:
  \[ t_{A \circ B} = \min(t_A, t_B) \]
An advanced solution

- require existence of an operator $\langle \cdot, \cdot \rangle$ with
  \[
  f : C \rightarrow \diamond A \quad g : C \rightarrow \diamond B
  \]
  \[
  \langle f, g \rangle : C \rightarrow \diamond (A \odot B)
  \]
- require $\odot$ to be a product functor in the Kleisli category of $\diamond$
- $\langle \cdot, \cdot \rangle$ is the $\langle \cdot, \cdot \rangle$-operator of $\odot$
- projections $\varpi_1$ and $\varpi_2$:
  - types:
    \[
    \varpi_1 : A \odot B \rightarrow \diamond A
    \]
    \[
    \varpi_2 : A \odot B \rightarrow \diamond B
    \]
  - types in verbose form:
    \[
    \varpi_1 : A \times B + A \times \diamond' B + \diamond' A \times B \rightarrow A + \diamond' A
    \]
    \[
    \varpi_2 : A \times B + A \times \diamond' B + \diamond' A \times B \rightarrow B + \diamond' B
    \]
  - definition of $\varpi_1$ and $\varpi_2$ is straightforward
The racing transformation in the advanced solution

- $r$ can be derived from $\langle \cdot, \cdot \rangle$:
  
  $$r := \langle \pi_1, \pi_2 \rangle$$

- Product axioms ensure that $r$ is an isomorphism with
  
  $$r^{-1} = \langle \mu(\Diamond \varpi_1), \mu(\Diamond \varpi_2) \rangle$$
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