Towards a Common Categorical Semantics for Linear-Time Temporal Logic and Functional Reactive Programming

Wolfgang Jeltsch

TTÜ Küberneetika Instituut
Tallinn, Estonia

28th Conference on the
Mathematical Foundations of Programming Semantics

June 9, 2012

Temporal logic

- intuitionistic temporal logic with temporal operators $\Box$ and $\Diamond$:

$$F ::= A \mid \top \mid \bot \mid F \wedge F \mid F \vee F \mid F \rightarrow F \mid \Box F \mid \Diamond F$$

- dependance on time:
  - time-dependent whether a proposition is true
  - time-dependent whether a certain proof proves a proposition
- meaning of $\Box$ and $\Diamond$:

$\Box\varphi$	φ holds at current and every future time
$\Diamond\varphi$	φ holds at current or some future time

Functional reactive programming

- functional programming with additional type constructors $\Box$ and $\Diamond$:

$$T ::= A \mid 1 \mid 0 \mid T \times T \mid T + T \mid T \to T \mid \Box T \mid \Diamond T$$

- dependance on time:
  - time-dependent whether a type is inhabited
  - time-dependent whether a certain value inhabits a type
- inhabitants of $\Box$ and $\Diamond$:

$\Box\tau$	time-varying value of type τ (behavior)
$\Diamond\tau$	time and associated value of type τ (event)

Simple categorical semantics

- ingredients of a categorical model:

  $(T, \leqslant)$ totally ordered set of times

  $\mathcal{C}$ bicartesian closed category (BCCC)

- $\mathcal{C}^T$ is a categorical model of temporal logic:

  object A maps times t to objects $A(t)$ of $\mathcal{C}$

  $f : A \to B$ maps times t to morphisms $f(t) : A(t) \to B(t)$

- endofunctors $\Box$ and $\Diamond$ defined as follows:

$$(\Box A)(t) := \prod_{t' \geqslant t} A(t') \qquad (\Diamond A)(t) := \coprod_{t' \geqslant t} A(t')$$

- possibly some infinite products and coproducts must exist in $\mathcal{C}$

Goal

axiomatic semantics that covers this semantics as a special case

Inspiration

Satoshi Kobayashi
Monad as Modality
Theoretical Computer Science 175 (1997), pp. 29–74

Gavin Bierman and Valeria de Paiva
On an Intuitionistic Modal Logic
Studia Logica 65 (2000), pp. 383–416

Basic structure

- bicartesian closed categories as the basis
- intuition of time independance:
  - $f : [\![\varphi]\!] \to [\![\psi]\!]$ models a proof showing that φ implies ψ at every time
  - $f : [\![\tau_1]\!] \to [\![\tau_2]\!]$ models a function from τ_1 to τ_2 that works at every time
- addition of endofunctors $\Box$ and $\Diamond$
- gives us functor applications:

$$\frac{f : A \to B}{\Box f : \Box A \to \Box B} \qquad \frac{f : A \to B}{\Diamond f : \Diamond A \to \Diamond B}$$

Monoidal functors

- $\Box$ is a strong monoidal functor on the cartesian structure (cartesian functor):

$$\Box A \times \Box B \cong \Box(A \times B)$$
$$1 \cong \Box 1$$

- $\Diamond$ is not a strong monoidal functor on the cocartesian structure:
    - natural transformations of these types would have to exist:

$$\Diamond(A + B) \to \Diamond A + \Diamond B$$
$$\Diamond 0 \to 0$$

    - correspond to non-causal functions in FRP

Comonads, monads, and tensorial strength

- $\Box$ is a comonad:

$$\varepsilon_A : \Box A \to A$$
$$\delta_A : \Box A \to \Box\Box A$$

- $\Diamond$ is a monad:

$$\eta_A : A \to \Diamond A$$
$$\mu_A : \Diamond\Diamond A \to \Diamond A$$

- $\Diamond$ is $\Box$-strong:

$$s_{A,B} : \Box A \times \Diamond B \to \Diamond(\Box A \times B)$$

Future only

- functors $\Box'$ and $\Diamond'$ with the following properties:

$$\Box A = A \times \Box' A$$
$$\Diamond A = A + \Diamond' A$$

- $\Box'$ is an ideal comonad:

$$\delta'_A : \Box' A \to \Box' \Box A$$

- $\Diamond'$ is an ideal monad:

$$\mu'_A : \Diamond' \Diamond A \to \Diamond' A$$

Linear time

- require existence of a natural transformation r with

$$r_{A,B} : \Diamond A \times \Diamond B \to \Diamond(A \odot B)$$

- definition of $\odot$:

$$A \odot B := A \times B + A \times \Diamond' B + \Diamond' A \times B$$

- alternatives of $A \odot B$ reflect relations between the time t_A of $\Diamond A$ and the time t_B of $\Diamond B$:

$A \times B$	$t_A = t_B$
$A \times \Diamond' B$	$t_A < t_B$
$\Diamond' A \times B$	$t_A > t_B$

- linearity of time is guaranteed:

$$(t_A = t_B) \vee (t_A < t_B) \vee (t_A > t_B)$$

- time of $\Diamond(A \odot B)$ is the minimum of the above times:

$$t_{A \odot B} = \min(t_A, t_B)$$

An advanced solution

- require existence of an operator $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ with

$$\frac{f : C \to \Diamond A \qquad g : C \to \Diamond B}{\langle\!\langle f, g \rangle\!\rangle : C \to \Diamond(A \odot B)}$$

- require $\odot$ to be a product functor in the Kleisli category of $\Diamond$
- $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is the $\langle \cdot, \cdot \rangle$-operator of $\odot$
- projections ϖ_1 and ϖ_2:
    - types:

$$\varpi_1 : A \odot B \to \Diamond A$$
$$\varpi_2 : A \odot B \to \Diamond B$$

    - types in verbose form:

$$\varpi_1 : A \times B + A \times \Diamond' B + \Diamond' A \times B \to A + \Diamond' A$$
$$\varpi_2 : A \times B + A \times \Diamond' B + \Diamond' A \times B \to B + \Diamond' B$$

    - definition of ϖ_1 and ϖ_2 is straightforward

The racing transformation in the advanced solution

- r can be derived from $\langle\!\langle \cdot, \cdot \rangle\!\rangle$:

$$r := \langle\!\langle \pi_1, \pi_2 \rangle\!\rangle$$

- product axioms ensure that r is an isomorphism with

$$r^{-1} = \langle \mu(\Diamond \varpi_1), \mu(\Diamond \varpi_2) \rangle$$

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