A Categorical Foundation of Functional Reactive Programming with Mutable State

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Linear functional programming

- values have to be used exactly once:
  - a value can represent the current state of an object
  - changes to the state (destructive updates) expressible as pure functions
  - some functions that describe destructive updates:

$$\begin{aligned} \mathit{newTmpFile} &: \mathit{World} \multimap \mathit{File} \otimes \mathit{World} \\ \mathit{writeNewLine} &: \mathit{File} \multimap \mathit{File} \\ \mathit{disposeTmpFile} &: \mathit{File} \otimes \mathit{World} \multimap \mathit{World} \end{aligned}$$

- Curry–Howard correspondent of intuitionistic linear logic
- modeled by symmetric monoidal closed categories (SMCCs)

Models of linear functional programming

- SMCC $(\mathcal{L}, \otimes, I, \multimap)$:
  - $\mathcal{L}$ for modeling types and linear functions
  - $\otimes$ for modeling linear pairs:

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$
$$A \otimes B \cong B \otimes A$$

  - I for modeling the linear unit:

$$I \otimes A \cong A$$
$$A \otimes I \cong A$$

  - $\multimap$ for modeling linear functions:

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, B \multimap C)$$

Models of non-linear functional programming

- cartesian closed category (CCC) $\mathcal{C}$:
  - $\mathcal{C}$ for modeling types and (non-linear) functions
  - $\times$ for modeling (non-linear) pairs:

$$\mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y) \cong \mathrm{Hom}(Z, X \times Y)$$

  - 1 for modeling the (non-linear) unit:

$$1 \cong \mathrm{Hom}(Z, 1)$$

  - $\Rightarrow$ for modeling (non-linear) functions:

$$\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, Y \Rightarrow Z)$$

- $(\mathcal{C}, \times, 1, \Rightarrow)$ is an SMCC with specific structure:

$$Z \to Z \times Z$$

$$Z \to 1$$

Interaction of non-linear and linear functional programming

- $(\mathcal{C}, \times, 1, \Rightarrow)$ and $(\mathcal{L}, \otimes, I, \multimap)$ connected by a lax symmetric monoidal adjunction (LSMA) $F \dashv G$:
  - F for using ordinary values in linear functional programming
  - G for modeling actions that create objects

$$\mathrm{Hom}(FX, A) \cong \mathrm{Hom}(X, GA)$$

$$FX \otimes FY \to F(X \times Y) \qquad GA \times GB \to G(A \otimes B)$$

$$I \to F1 \qquad 1 \to GI$$

- implies isomorphisms:

$$FX \otimes FY \cong F(X \times Y)$$

$$I \cong F1$$

- see the work of Benton (1994)

Functional reactive programming (FRP) and its models

- FRP captures temporal aspects of programs:
  - contains constructs that describe temporal behavior
  - type-inhabitance is time-dependent
  - in general, times can be any totally ordered set
  - here restriction to discrete time
- Curry–Howard correspondent of intuitionistic temporal logic
- modeled by CCCs $\mathcal{T}$ with a cartesian endofunctor $\bigcirc$:

$\mathcal{T}$	for modeling types and time-universal functions
$\times$, 1, $\Rightarrow$	for modeling pairs, the unit, and functions
$\bigcirc$	for modeling values at the next time:

$$\bigcirc A \times \bigcirc B \cong \bigcirc(A \times B)$$

$$1 \cong \bigcirc 1$$

Interaction of functional programming and FRP

- $\mathcal{C}$ and $\mathcal{T}$ connected by an adjunction $F \dashv G$:
  - F for modeling types with time-independent type inhabitance
  - G for modeling time-universal values

$$\mathrm{Hom}(FX, A) \cong \mathrm{Hom}(X, GA)$$

- require that adjunction interacts sensibly with "next" functors:

$$F(\bigcirc X) \to \bigcirc(FX)$$
$$\bigcirc(GA) \to G(\bigcirc A)$$

- for the CCC $\mathcal{C}$ (which models functional programming), we take the identity functor as a specific "next" functor:

$$FX \to \bigcirc(FX)$$
$$GA \to G(\bigcirc A)$$

Unifying the linear and the temporal adjunction

- what we have so far:
  - $(\mathcal{C}, \times, 1, \Rightarrow)$ and $(\mathcal{L}, \otimes, I, \multimap)$ connected by an LSMA
  - $(\mathcal{C}, \mathrm{Id})$ and $(\mathcal{T}, \bigcirc)$ connected by an adjunction that "respects" cartesian endofunctors
- generalize cartesian endofunctors to symmetric monoidal endofunctors (SMEs):
  - work also in a linear setting, where we do not have products in general
  - for $\mathcal{L}$, we use the identity functor, like we did for $\mathcal{C}$
- now we require both adjunctions to be LSMAs and to respect SMEs
- makes both adjoint functors of the $\mathcal{C}$–$\mathcal{T}$ adjunction cartesian:

$$FX \times FY \cong F(X \times Y) \qquad GA \times GB \cong G(A \times B)$$

$$1 \cong F1 \qquad 1 \cong G1$$

Adding models of linear FRP

- consider the following category:

  objects SMCCs with an SME

  morphisms LSMAs that respect SMEs

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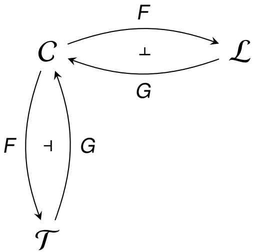
- the unified adjunctions from the last slide form a span in this category



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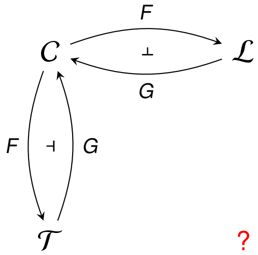

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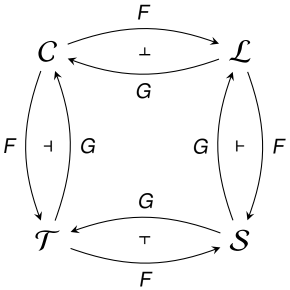

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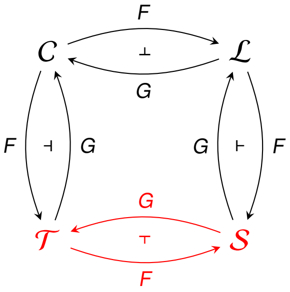
- the unified adjunctions from the last slide form a span in this category
- is there a category that models a linear variant of FRP?
- define it to be the pushout of the span
- interaction of $\mathcal{T}$–$\mathcal{S}$ adjunction with $\bigcirc$ makes sense:

$$F(\bigcirc X) \to \bigcirc(FX)$$
$$\bigcirc(GA) \to G(\bigcirc A)$$



Outlook

- extend this work to continuous time:
  - categorical models of FRP that also cover continuous time exist
  - are based on process functor $\triangleright''$
  - interaction of adjunctions with $\triangleright''$ unclear in general
  - in the discrete case, $\triangleright''$ can be defined in terms of $\bigcirc$
  - idea: make the discrete case continuous by using hyperreal numbers (see Beauxis and Mimram 2011)
- turn the semantics into an API of an FRP library

References

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