Temporal Logic with "Until", Functional Reactive Programming with Processes, and Concrete Process Categories

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Functional reactive programming

- extension of functional programming
- allows programs to deal with temporal aspects in a declarative fashion
- two key concepts:
  - time-dependent type membership
  - special type constructors:
    - $\Box$ time-varying values
    - $\Diamond$ events
- Curry–Howard correspondence to temporal logic:
  - time-dependent trueness
  - special operators:
    - $\Box$ will always hold
    - $\Diamond$ will eventually hold

Categorical models of FRP

- ingredients:

  totally ordered set $(T, \leqslant)$ time scale

  CCCC $\mathcal{B}$ simple types and functions
- functor category $\mathcal{B}^T$ models FRP types and operations with indices denoting inhabitation times
- CCCC structure of $\mathcal{B}^T$ from CCCC structure of $\mathcal{B}$ with operations working pointwise
- functors $\Box$ and $\Diamond$ defined as follows:

$$(\Box A)(t) = \prod_{t' \geqslant t} A(t')$$

$$(\Diamond A)(t) = \coprod_{t' \geqslant t} A(t')$$

From "until" to processes

- more temporal operators from linear-time temporal logic:
  - $\triangleright'$ strong "until"
  - $\blacktriangleright'$ weak "until"
- semantics given by functors $\triangleright'$ and $\blacktriangleright'$:

$$(A \triangleright' B)(t) = \coprod_{t' \in (t,\infty)} \left(\left(\prod_{t'' \in [t,t')} A(t'') \right) \times B(t') \right)$$

$$(A \blacktriangleright' B)(t) = (A \triangleright' B)(t) + \prod_{t' \in [t,\infty)} A(t')$$

- FRP analogs of "until" proofs are processes:
  - normally finite-length time-varying value plus terminal event
  - in the case of $\blacktriangleright'$ also nontermination possible

Applications of processes

- stereo playback with different guarantees:

  $(\mathbb{R} \times \mathbb{R}) \blacktriangleright' 1$ none

  $(\mathbb{R} \times \mathbb{R}) \triangleright' 1$ termination

  $(\mathbb{R} \times \mathbb{R}) \blacktriangleright' 0$ nontermination

- stereo playback with additional information:

  $(\mathbb{R} \times \mathbb{R}) \blacktriangleright' (1 + 1)$ reason of termination (end of track vs. abort)

- alternating stereo/mono playback with different guarantees:

  $\nu\alpha \,.\, (\mathbb{R} \times \mathbb{R}) \blacktriangleright' \mathbb{R} \blacktriangleright' \alpha$ nontermination

  $\nu\alpha \,.\, (\mathbb{R} \times \mathbb{R}) \triangleright' \mathbb{R} \triangleright' \alpha$ switch, nontermination

  $\nu\alpha \,.\, (\mathbb{R} \times \mathbb{R}) \blacktriangleright' (1 + \mathbb{R} \blacktriangleright' (1 + \alpha))$ none

  $\nu\alpha \,.\, (\mathbb{R} \times \mathbb{R}) \triangleright' (1 + \mathbb{R} \triangleright' (1 + \alpha))$ switch

  $\mu\alpha \,.\, (\mathbb{R} \times \mathbb{R}) \triangleright' (1 + \mathbb{R} \triangleright' (1 + \alpha))$ termination

Processes as the core concept of FRP

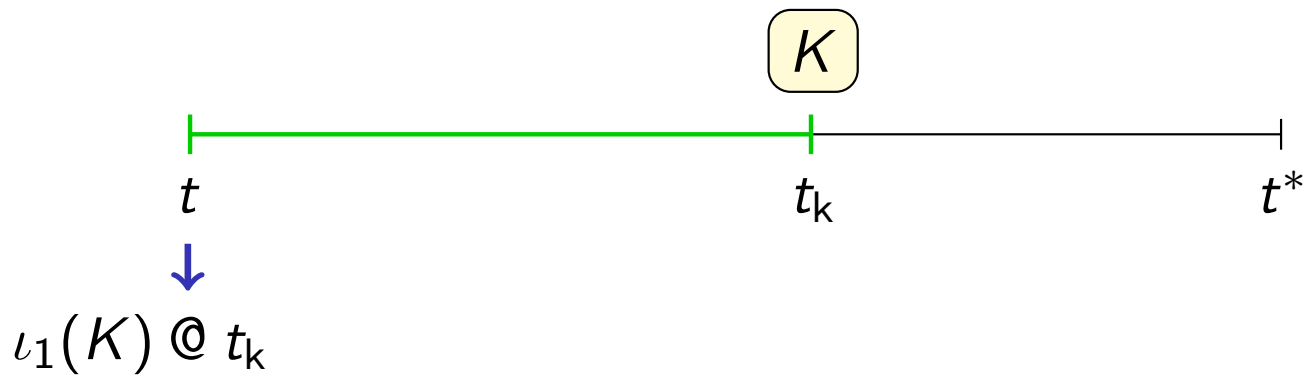
- introduction of processes increases expressiveness
- processes cover time-varying values and events as special cases:

$$\Box A \cong A \blacktriangleright' 0$$
$$\Diamond A \cong A + 1 \triangleright' A$$

An example program component

- looks for the next key press up to a certain timeout
- emits a value of type $\Diamond(\mathrm{Key} + 1)$ when it starts:

  Case 1 key press before timeout:



  Case 2 no key press before timeout:

A noncausal operation

- hypothetical polymorphic operation d from $\Diamond(\tau_1 + \tau_2)$ to $\Diamond\tau_1 + \Diamond\tau_2$:

$$\iota_1(x) \mathbin{@} t' \mapsto \iota_1(x \mathbin{@} t')$$
$$\iota_2(y) \mathbin{@} t' \mapsto \iota_2(y \mathbin{@} t')$$

- applying d to the output of the key press listener gives value of type $\Diamond\mathsf{Key} + \Diamond 1$:

  key press before timeout $\iota_1(K \mathbin{@} t_\mathsf{k})$
  no key press before timeout $\iota_2(\mathsf{tt} \mathbin{@} t^*)$

- tells us immediately if the user will press a key before the timeout
- so d cannot exist

Semantics allow for noncausal operations

- polymorphic operations from $\Diamond(\tau_1 + \tau_2)$ to $\Diamond\tau_1 + \Diamond\tau_2$ modeled by natural transformations τ with

$$\tau_{A,B} : \Diamond(A + B) \to \Diamond A + \Diamond B$$

- there is such a τ (which is even an isomorphism):

$$\coprod_{t' \geqslant t} \Big(A(t') + B(t') \Big) \cong \coprod_{t' \geqslant t} A(t') + \coprod_{t' \geqslant t} B(t')$$

- reason:

  semantics do not deal with time-dependent knowledge about values

Knowledge-aware semantics

- replace category $\mathcal{B}^T$ by category $\mathcal{B}^I$ where

$$I = \{(t, t_o) \in T \times T \mid t \leqslant t_o\}$$

- dealing with knowledge at t_o:
  - objects $A(t, t_o)$ deal with knowledge at t_o
  - morphisms $f_{(t,t_o)}$ transform knowledge
- $(A \blacktriangleright' B)(t, t_o)$ defined as follows:

$$\coprod_{t' \in (t,t_o]} \left(\left(\prod_{t'' \in [t,t')} A\left(t'', t_o\right) \right) \times B\left(t', t_o\right) \right) + \prod_{t' \in [t,t_o]} A\left(t', t_o\right)$$

Compatibility of knowledge transformations

- knowledge transformations $f(t, t_o)$ and $f(t, t'_o)$ must be compatible
- extend set I to category $\mathcal{I}$ by adding morphisms

$$(t, t_o, t'_o) : (t, t'_o) \to (t, t_o)$$

for $t \leqslant t_o \leqslant t'_o$

- modeling of knowledge reduction:

$$A(t, t_o, t'_o) : A(t, t'_o) \to A(t, t_o)$$

- morphisms of $\mathcal{B}^{\mathcal{I}}$ are natural transformations:

$$\begin{array}{ccc} A(t, t_o) & \xleftarrow{A(t, t_o, t'_o)} & A(t, t'_o) \\ \Big\downarrow{\scriptstyle f_{(t,t_o)}} & & \Big\downarrow{\scriptstyle f_{(t,t'_o)}} \\ B(t, t_o) & \xleftarrow[B(t, t_o, t'_o)]{} & B(t, t'_o) \end{array}$$

Upper bounds for occurrence times

- definition of functor $\rhd'$ not directly possible
- introduction of new functor $\rhd'_{\_} : \mathcal{T} \to \left(\mathcal{B}^{\mathcal{I}}\right)^{\mathcal{B}^{\mathcal{I}} \times \mathcal{B}^{\mathcal{I}}}$ where $\mathcal{T}$ is the category of $(T, \leqslant)$
- $\rhd'_{t_\mathrm{b}}$ models a process type constructor with upper bound t_b for termination time
- $(A \rhd'_{t_\mathrm{b}} B)(t, t_\mathrm{o})$ defined as follows:

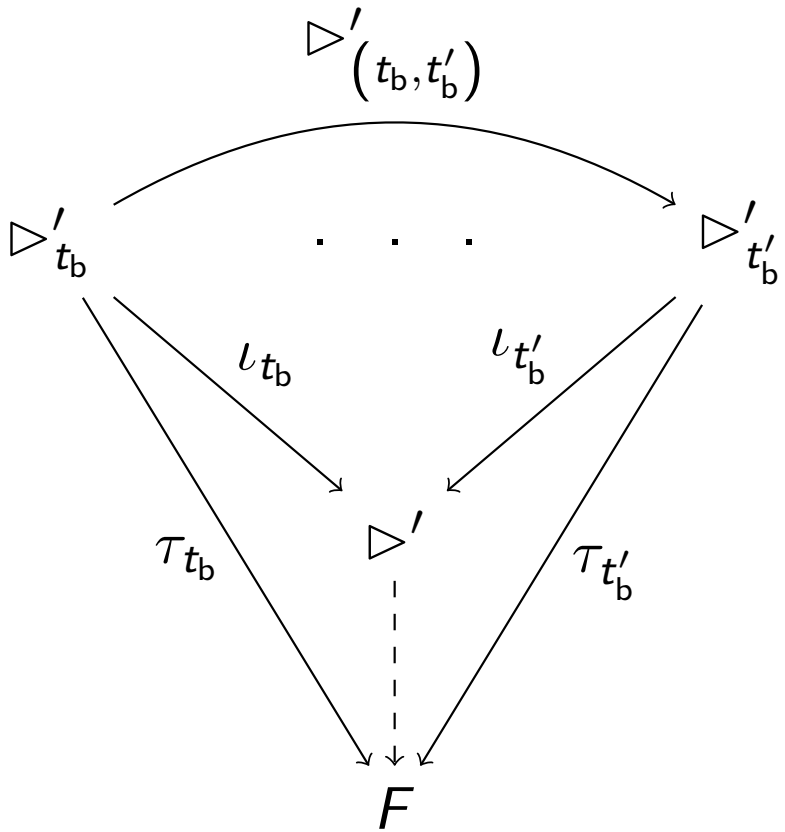
$$\begin{cases} 0 & \text{if } t_\mathrm{b} < t \\ \coprod_{t' \in (t, t_\mathrm{b}]} \left(\left(\prod_{t'' \in [t, t')} A(t'', t_\mathrm{o}) \right) \times B(t', t_\mathrm{o}) \right) & \text{if } t \leqslant t_\mathrm{b} \leqslant t_\mathrm{o} \\ (A \blacktriangleright' B)(t, t_\mathrm{o}) & \text{it } t_\mathrm{o} < t_\mathrm{b} \end{cases}$$

- modeling of type conversion:

$$\rhd'_{\left(t_\mathrm{b}, t'_\mathrm{b}\right)} : \rhd'_{t_\mathrm{b}} \to \rhd'_{t'_\mathrm{b}}$$

Definition of the $\triangleright'$-functor

- type constructor $\triangleright'$ is the least upper bound of all $\triangleright'_{t_b}$-constructors
- functor $\triangleright'$ must be a colimit of the functor $\triangleright'\_$:



The shape of the $\triangleright'$-functor

Theorem

If $(T, \leqslant)$ has a maximum $t_{\max}$, then $\triangleright' \cong \triangleright'_{t_{\max}}$.

Theorem

If $(T, \leqslant)$ has no maximum, then $\triangleright' \cong \blacktriangleright'$.

Causality ensured

Theorem

There are concrete process categories that do not contain any natural transformation τ with

$$\tau_{A,B} : \Diamond(A + B) \to \Diamond A + \Diamond B \quad .$$

Conclusions

- processes:
  - result of extending the Curry–Howard correspondence between FRP and temporal logic to cover "until" operators
  - make FRP more expressive
  - generalize time-varying values and events nicely
- knowledge-aware categorical models:
  - express causality of FRP operations
  - cannot express liveness constraint of $\triangleright'$ for unbounded time
- ultimate goal is an axiomatic semantics with the following properties:
  - expresses causality
  - expresses liveness constraint of $\triangleright'$ in general
  - covers concrete process categories as a special case